Estimating a VECM for a small open economy

Sebastian Ankargren and Johan Lyhagen

Department of Statistics, Uppsala University

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Abstract

In economic theory, the term *small open economy* refers to an economy that is too small to influence the surrounding world. The surrounding world can, for this reason, be seen as exogenous relative to the economy of this small open economy. The main contribution of this paper is the proposal of how to estimate a vector error correction model with exogeneity restrictions on the long-run and short-run adjustment parameters as well as on the short-run dynamic parameters between small open economies and the surrounding world. A Monte Carlo simulation study of impulse responses shows that the proposed method is considerably more efficient compared to models that fully or partially ignore the restrictions implied by the small open economy hypothesis. Using two Swedish macroeconomic datasets, we find that there are, for some variables, large differences in impulse responses between our proposed method incorporating the restrictions and models using no or partial restrictions. As the small open economy hypothesis is in many situations uncontroversial, our method enables the incorporation of indisputable economic theory into the econometric estimation of the model.

Keywords: Granger non-causality; Impulse responses; Strong exogeneity; Cointegration.

1 Introduction

A small open economy is an economy that is too small to influence the surrounding world. For example, a small open economy does not influence global prices, interest rates or economic conditions. The small open economy label applies to most countries today except for leading economies such as the United States and China, or regions like the EU in which the countries grouped together no longer constitute a small open economy. Hence, there is a natural role for developing economic theories that apply to small open economies.

Statistics has played a crucial role in economics since the seminal paper of Haavelmo (1944), which lay the statistical foundation for applied macroeconomics and theoretically justified the work of the Cowles commission. The main focus when modeling macroeconomic relations was, at the time, on large systems of equations and the majority of the work in macroeconometrics focused on related issues such as identification, endogeneity, system estimation, etc. Focus shifted when Sims (1980) criticized large-scale structural econometric models and popularized the vector autoregressive (VAR) approach. An important contribution of the Sims (1980) paper is the illustration of the usefulness of impulse response analysis.

Engle and Granger (1987) introduced cointegration, which is a concept that enables modeling of economic equilibria. The main breakthrough was Johansen's maximum likelihood approach to cointegration (see e.g. Johansen 1988, 1991, 1995) that effectively merged the VAR model with the concept of cointegration. This synthesis made it possible to test economic theories through tests involving the cointegrating relations and stochastic trends, but also for policy evaluations using impulse response analysis. Using Johansen's approach it is possible to estimate and test restrictions on the cointegrating, or equilibrium, relations, as well as on the adjustment parameters. Johansen proposed the use of reduced rank regression to estimate the parameters of the cointegrating relations and the adjustment parameters. To solve the problem with short-run dynamics, the first step of the procedure consists of using the Frisch-Waugh-Lovell theorem to concentrate out the short-run dynamics.

When employing the model for a small open economy it is customary to have two sets of variables. The first set consists of the domestic variables of interest (e.g. GDP, exports, imports and inflation) and the second set contains foreign variables (such as foreign GDP and interest rates). It is important to notice that the concept of a small open economy implies that there is no feedback from the small economy to the foreign economy. If this exogeneity is taken into account in the model it is usually accomplished by restricting the appropriate adjustment parameters to zero, but it may simply also be ignored. The standard Johansen approach concentrates out the short-run dynamics, but doing so is not a viable option when exogeneity is imposed as the short-run dynamics involving the domestic variables are no longer common to all variables and hence cannot be concentrated out. Therefore, restrictions on the short-run dynamics are relatively scarce in the literature as estimation is somewhat more involved. Asymptotically, however, neglecting to impose small open economy exogeneity does not affect the properties of the estimator of the cointegrating vectors.

Our situation of modeling a small open economy aligns with the more general situation of Granger non-causality in VECMs. Granger non-causality describes the situation when one set of variables lacks predictive information about another set of variables. Granger non-causality and tests for its presence have been studied by, e.g., Mosconi and Giannini (1992); Rault (2000) and Ahn et al. (2015) recently developed an estimator based on the approach proposed by Ahn and Reinsel (1990).

Our contribution consists of two parts. First, we develop an estimator for vector error correction models for small open economies. The estimator is a switching estimator based on the work by Boswijk (1995) and Groen and Kleibergen (2003) and we provide accompanying asymptotic distribution theory for the estimator. Second, it is well-known that the maximum likelihood estimator for the cointegrating vectors is asymptotically independent of the estimator for the adjustment parameters. Therefore, asymptotic inference for the cointegrating vectors can be conducted without respect to the other parameters of the model, and because the restricted model is nested in the unrestricted model estimation is also consistent. However, our primary interest is not the cointegrating vectors; instead, as VARs are often used for impulse response analysis we study the loss in precision incurred by neglecting exogeneity in the model in finite-sample situations. We conduct a simulation study and compare impulse responses in two empirical applications to shed light on the effect the failure to acknowledge exogeneity may have on subsequent impulse response analysis. The results show that the impulse responses can differ notably if the correct exogeneity structure is not imposed.

The paper is organized as follows. The next section introduces the model and the main restrictions of interest as well as the estimation procedure. Section 3 analyzes the effect of imposing the restrictions by Monte Carlo simulation while two empirical examples are discussed in Section 4. A conclusion ends the paper.

2 Model and Estimation

The vector error correction model (VECM) for the $k \times 1$ vector y_t can be written as

$$\Delta y_t = \alpha\beta' y_{t-1} + \sum_{i=1}^{p} \Gamma_i \Delta y_{t-i} + \varepsilon_t \quad t = 1, \ldots, T, \tag{1}$$

where α and β are full column rank matrices of size $k \times r$, Γ_i are $k \times k$ matrices containing the short-run dynamics parameters, ε_t is a $k \times 1$ vector of white noise disturbances with covariance matrix Ω, and $\Delta y_t = y_t - y_{t-1}$. We exclude deterministic terms to keep the exposition simple, but note that these can easily be incorporated. The vector y_t is assumed to be integrated of order one (Johansen, 1995), implying that both the first difference Δy_t and the linear combination $\beta' y_t$ are stationary with the latter describing economic equilibria. The matrix α describes the speed of adjustment towards the equilibria defined by the cointegrating relations.

We develop a switching estimator that builds on the methods developed by Boswijk (1995); Groen and Kleibergen (2003). To this end, let $y_t' = \left[y_{f,t}', y_{d,t}'\right]'$ where $y_{f,t}$ is the set of s (exogenous) variables from the foreign economy and $y_{d,t}$ the $k - s$ (endogenous) domestic variables. A small open economy paradigm implies that $\Delta y_{d,t-1}, \ldots, \Delta y_{d,t-p}$ should not affect $\Delta y_{f,t}$ and as such justifies the restricted model

$$\begin{aligned} \begin{bmatrix} \Delta y_{f,t} \\ \Delta y_{d,t} \end{bmatrix} &= \alpha\beta' \begin{bmatrix} y_{f,t-1} \\ y_{d,t-1} \end{bmatrix} + \sum_{i=1}^{p} \Gamma_i \begin{bmatrix} \Delta y_{f,t-i} \\ \Delta y_{d,t-i} \end{bmatrix} + \varepsilon_t \\ &= \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1' & \beta_2' \end{bmatrix} \begin{bmatrix} y_{f,t-1} \\ y_{d,t-1} \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} \Gamma_{11i} & 0 \\ \Gamma_{21i} & \Gamma_{22i} \end{bmatrix} \begin{bmatrix} \Delta y_{f,t-i} \\ \Delta y_{d,t-i} \end{bmatrix} + \varepsilon_t, \end{aligned} \tag{2}$$

where α_1 and β_1 are $s \times r$ and α_2 and β_2 are $(k - s) \times r$.

The blocks of α and β are left unrestricted for the moment; we will return to them in the next subsection. To estimate the model, we first note that as the vector $\Delta y_{f,t-i}$ influences all left-hand side variables we can use the Frisch-Waugh-Lovell theorem to concentrate them out. Let $\Delta\tilde{y}_t$, $\tilde{y}_{t-1}$, $\Delta\tilde{y}_{d,t-i}$ and $\tilde{\varepsilon}_t$ denote the remaining variables with $\Delta y_{f,t-i}$ partialed out. Stacking of the observations yields the model in matrix form

$$\tilde{Y} = \tilde{Y}_{-1}\Pi + \tilde{\varepsilon} \tag{3}$$

where

$$\Pi = \begin{bmatrix} \multicolumn{2}{c}{\beta\alpha'} \\ 0' & \Gamma'_{221} \\ \vdots & \vdots \\ 0' & \Gamma'_{22p} \end{bmatrix} \tag{4}$$

and

$$\tilde{Y} = \begin{bmatrix} \Delta\tilde{y}'_1 \\ \Delta\tilde{y}'_2 \\ \vdots \\ \Delta\tilde{y}'_T \end{bmatrix}, \tilde{Y}_{-1} = \begin{bmatrix} \tilde{y}'_0 & \Delta\tilde{y}'_{d,-1} & \cdots & \Delta\tilde{y}'_{d,-p} \\ \tilde{y}'_1 & \Delta\tilde{y}'_{d,0} & & \Delta\tilde{y}'_{d,-p+1} \\ \vdots & & \ddots & \\ \tilde{y}'_{T-1} & \Delta\tilde{y}'_{d,T-1} & & \Delta\tilde{y}'_{d,T-p} \end{bmatrix}. \tag{5}$$

The concentrated log-likelihood is, up to a constant,

$$\ell(\Pi, \Omega) = -\frac{T}{2}|\Omega| - \frac{1}{2}\operatorname{vec}(\tilde{Y} - \tilde{Y}_{-1}\Pi)'(\Omega^{-1} \otimes I_T)\operatorname{vec}(\tilde{Y} - \tilde{Y}_{-1}\Pi). \tag{6}$$

The second term in the above display can be rewritten into

$$G(\Pi, \Omega) = \operatorname{vec}\left[\tilde{Y}'_{-1}\left(\tilde{Y} - \tilde{Y}_{-1}\Pi\right)\right]'\left[\Omega \otimes \left(\tilde{Y}'_{-1}\tilde{Y}_{-1}\right)\right]^{-1}\operatorname{vec}\left[\tilde{Y}'_{-1}\left(\tilde{Y} - \tilde{Y}_{-1}\Pi\right)\right], \tag{7}$$

where maximization of (6) is equivalent to minimization of (7). For the time being, we assume Ω is known. To accommodate minimization of (7), we first rewrite $\operatorname{vec}\left[\tilde{Y}'_{-1}\left(\tilde{Y} - \tilde{Y}_{-1}\Pi\right)\right]$ as

$$\begin{aligned} \operatorname{vec}\left[\tilde{Y}'_{-1}\left(\tilde{Y} - \tilde{Y}_{-1}\Pi\right)\right] &= \operatorname{vec}\left(\tilde{Y}'_{-1}\tilde{Y}\right) - \operatorname{vec}\left(\tilde{Y}'_{-1}\tilde{Y}_{-1}\Pi\right) &(8)\\ &= \operatorname{vec}\left(\tilde{Y}'_{-1}\tilde{Y}\right) - F\pi, &(9) \end{aligned}$$

where $F = \left(I_k \otimes \tilde{Y}'_{-1}\tilde{Y}_{-1}\right)$ and $\pi = \operatorname{vec}(\Pi)$. The underlying idea for the estimation procedure is that $F\pi$ can, to begin with, be written in two different ways, each conditional on one of α and β. To fix ideas, first let

$$\pi_\beta = \begin{bmatrix} \operatorname{vec}(\beta') \\ \operatorname{vec}(\Gamma_{221}) \\ \vdots \\ \operatorname{vec}(\Gamma_{22p}) \end{bmatrix}, \quad \pi_\alpha = \begin{bmatrix} \operatorname{vec}(\alpha) \\ \operatorname{vec}(\Gamma_{221}) \\ \vdots \\ \operatorname{vec}(\Gamma_{22p}) \end{bmatrix} \tag{10}$$

Then, the generic $F\pi$ can be written as

$$\begin{aligned} F\pi &= F_\beta(\alpha)\pi_\beta &(11)\\ &= F_\alpha(\beta)\pi_\alpha &(12) \end{aligned}$$

where

$$F_\beta(\alpha) = \left(I_k \otimes \tilde{Y}_{-1}'\tilde{Y}_{-1}\right) K_{k,k+p(k-s)} \begin{bmatrix} I_k \otimes \underset{k\times r}{\alpha} & \underset{k^2\times p(k-s)^2}{0} \\ \underset{pk(k-s)\times kr}{0} & I_{p(k-s)} \otimes \begin{bmatrix} 0_{s\times k-s} \\ I_{k-s} \end{bmatrix} \end{bmatrix} \tag{13}$$

and

$$F_\alpha(\beta) = \left(I \otimes \tilde{Y}_{-1}'\tilde{Y}_{-1}\right) K_{k,k+p(k-s)} \begin{bmatrix} \underset{k\times r}{\beta} \otimes I_k & \underset{k^2\times p(k-s)^2}{0} \\ \underset{pk(k-s)\times kr}{0} & I_{p(k-s)} \otimes \begin{bmatrix} 0_{s\times k-s} \\ I_{k-s} \end{bmatrix} \end{bmatrix}. \tag{14}$$

We are here using $K_{m,n}$ to denote the commutation matrix for an $m \times n$ matrix A defined by $K_{m,n} \operatorname{vec}(A) = \operatorname{vec}(A')$ (Magnus and Neudecker, 1979).

As is clear from (10), estimating π_α and π_β means estimating the full $k \times r$ matrices α and β without restrictions. However, when the model is used for modeling a small open economy where exogenous, large-economy variables are also included such an approach may not be reasonable. If α and β are left unrestricted then the model will allow for the r cointegrating relations $\beta' y_{t-1}$ to enter all equations in the system, including the equations for the foreign variables. As a small open economy should not be able to have this influence, shutting down such a connection is in many modeling situations warranted and uncontroversial. Inspired by our two empirical applications, we consider the setting of Granger non-causality (Mosconi and Giannini, 1992; Rault, 2000) for enforcing the small open economy property in our model. Granger non-causality directly includes weak and strong exogeneity (Johansen, 1992; Harbo et al., 1998; Pesaran et al., 2000; Jacobs and Wallis, 2010) as a special case, but situations involving super exogeneity (Pradel and Rault, 2003) and cointegrating exogeneity (Hunter, 1992) can just as well be accomodated by judicious construction of the F matrix. Independently of our work and with a different focus, Ahn et al. (2015) developed an estimation procedure based on the reparametrization technique used by Ahn and Reinsel (1990).

2.1 The Small Open Economy Property: Granger Non-Causality Restrictions

As a first example of restrictions stemming from the small open economy property, we can assume that the foreign variables are weakly exogenous for the cointegrating vectors β such that $\alpha_2 = 0_{s\times r}$. Exogeneity as discussed by Engle et al. (1983) is defined with respect to a set of parameters that are of primary interest. The cointegrating vectors β are often the primary interest in studies using VECMs, and so since β only enters the domestic set of equations it suffices to analyze this subsystem, the conditional model, instead of the full model. However, our interest lies in impulse response analysis in cointegrated small open economy VARs in which case the full set of parameters is needed. The foreign variables are clearly not weakly exogenous for the full set of parameters, and analysis of a partial system is not feasible. For further discussion on partial systems and weak exogeneity, see Johansen (1992); Harbo et al. (1998); Jacobs and Wallis (2010).

Assuming weak exogeneity is in many cases overly restrictive as it excludes any cointegrating relation from entering the foreign equations. Thereby, cointegration among the foreign variables is prohibited. A more flexible scheme that still enforces the small open economy constraints is that of Granger non-causality; see for example Mosconi and Giannini (1992); Toda and Phillips (1993); Rault (2000) for general discussions of Granger non-causality. Suppose that there are r_1 foreign and r_2 domestic cointegrating relations with $r_1 + r_2 = r$. If we partition α and β into blocks with r_1 and r_2 columns and s and $k - s$ rows we have

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}. \tag{15}$$

Granger non-causality enforces the joint restriction that

$$\alpha_{12} = 0_{s \times r_2}, \quad \beta_{21} = 0_{k-s \times r_1}, \quad \Gamma_{12i} = 0_{s \times k-s}, \tag{16}$$

see also Mosconi and Giannini (1992); Ericsson et al. (1998). The matrix $\Pi = \alpha\beta'$ is block triangular under the restrictions in (16) and parallels the notion of cointegrating exogeneity (Hunter, 1992), but without the further restriction on Γ_{12i} the domestic set of variables will still Granger-cause the foreign block. In one of our applications later, we will revisit a model for the Swedish economy in which there are four domestic and three foreign variables. In this case, two cointegrating relations among all variables exclusively enter the domestic part of the system and one cointegrating relation involving only foreign variables is allowed to enter all equations. For situations when the cointegrating rank is unknown, the reparametrization and testing procedure suggested by Rault (2000) can be employed.

As a special case, consider the situation when $r_1 = 0$. Then $r_2 = r$ and $\alpha = \begin{bmatrix} 0_{r \times s} & \alpha'_{22} \end{bmatrix}'$, which corresponds to the previously mentioned weak exogeneity restriction. However, since $\Gamma_{12i} = 0_{s \times k-s}$ is also imposed, the foreign variables are strongly exogenous for the parameters in the domestic submodel since both weak exogeneity and Granger non-causality are present (Engle et al., 1983, Definition 2.6). Strong exogeneity no longer holds when the foreign variables cointegrate (i.e. when α_{11} is a non-zero and non-empty matrix) as β'_1 is then included in both the domestic and foreign submodels.

2.2 Estimation

Estimation is carried out by constructing the F matrices as

$$\begin{aligned} F_\beta^{(GN)}(\alpha) &= \left(I_k \otimes \tilde{Y}'_{-1}\tilde{Y}_{-1}\right) K_{k,k+p(k-s)} \\ &\times \left[\begin{array}{c|c|c} \begin{bmatrix} I_s \\ 0_{k-s \times s} \end{bmatrix} \otimes \alpha & \begin{matrix} 0_{ks \times r_2(k-s)} \\ I_{k-s} \otimes \alpha \begin{bmatrix} 0_{r_1 \times r_2} \\ I_{r_2} \end{bmatrix} \end{matrix} & 0_{r(k-s) \times p(k-s)^2} \\ \hline 0_{pk(k-s) \times rs} & 0_{pk(k-s) \times r_2(k-s)} & I_{p(k-s)} \otimes \begin{bmatrix} 0_{s \times k-s} \\ I_{k-s} \end{bmatrix} \end{array}\right] \end{aligned} \tag{17}$$

$$\begin{aligned} F_\alpha^{(GN)}(\beta) &= \\ F_\alpha(\beta) &\left[\begin{array}{c|c|c} K_{r,k} \begin{bmatrix} I_s \otimes \begin{bmatrix} I_{r_1} \\ 0_{r_2 \times r_1} \end{bmatrix} \\ 0_{r(k-s) \times r_1 s} \end{bmatrix} K_{s,r_1} & I_r \otimes \begin{bmatrix} 0_{s \times k-s} \\ I_{k-s} \end{bmatrix} & 0_{rk \times p(k-s)^2} \\ \hline 0_{p(k-s)^2 \times r_1 s} & 0_{p(k-s)^2 \times r(k-s)} & I_{p(k-s)^2} \end{array}\right] \end{aligned}$$

where the parameter vectors are

$$\pi_\beta^{(GN)} = \begin{bmatrix} \text{vec}(\beta_1') \\ \text{vec}(\beta_{22}') \\ \text{vec}(\Gamma) \end{bmatrix} \quad \pi_\alpha^{(GN)} = \begin{bmatrix} \text{vec}(\alpha_{11}) \\ \text{vec}(\alpha_2) \\ \text{vec}(\Gamma) \end{bmatrix}. \tag{18}$$

By substituting vec $\left[\tilde{Y}_{-1}'\left(\tilde{Y} - \tilde{Y}_{-1}\Pi\right)\right]$ for (9) and $F\pi$ for the desired choice of F and π it is possible to solve for π_β and π_α, respectively. The solutions are

$$\hat{\pi}_\beta(\alpha, \Omega) = \left\{F_\beta(\alpha)'\left[\Omega \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right]^{-1} F_\beta(\alpha)\right\}^{-1} F_\beta(\alpha)' \tag{19}$$

$$\times \left[\Omega \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right]^{-1} \text{vec}\left(\tilde{Y}_{-1}'\tilde{Y}\right) \tag{20}$$

$$\hat{\pi}_\alpha(\beta, \Omega) = \left\{F_\alpha(\beta)'\left[\Omega \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right]^{-1} F_\alpha(\beta)\right\}^{-1} F_\alpha(\beta)' \tag{21}$$

$$\times \left[\Omega \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right]^{-1} \text{vec}\left(\tilde{Y}_{-1}'\tilde{Y}\right) \tag{22}$$

Enforcing Granger non-causality simply amounts to substituting the F matrices yielding the unrestricted estimates with $F^{(GN)}$ in (19)–(21); in a similar fashion, other forms of restricted models discussed in the beginning of the section can be estimated using the same approach by constructing the F matrix appropriately.

Finally, rarely ever is Ω a known matrix and it too must be estimated. To facilitate this, we use the conditional maximum likelihood estimator of Ω conditional on Π given by

$$\hat{\Omega}(\Pi) = \frac{1}{T}(Y - Y_{-1}\Pi)'(Y - Y_{-1}\Pi) \tag{23}$$

Evidently, there is a circular dependence in the equations which implicitly suggests an iterative estimation procedure. This iterative procedure is as follows:

1. Estimate Ω, α in an unrestricted VECM
2. Estimate π_β using $\hat{\pi}_\beta(\hat{\alpha}, \hat{\Omega})$ in (19)
3. Estimate π_α using $\hat{\pi}_\alpha(\hat{\beta}, \hat{\Omega})$ in (21)
4. Estimate Ω using $\hat{\Omega}(\hat{\Pi})$ in (23)
5. Iterate 2–4 until convergence

Such a switching algorithm has previously been applied in the cointegration literature by e.g. Johansen and Juselius (1992, 1994); Groen and Kleibergen (2003); Boswijk and Doornik (2004). While none of the previous studies have proven that the algorithm converges to a global maximum, each step is non-decreasing in the likelihood and generally works very well. To start the iterative procedure, we have used the unrestricted estimates and found this approach to work well in practice. The asymptotic properties of the estimator based on (17)–(18) are established in the following proposition in which we restrict ourselves to the case without deterministic terms for simplicity, but without loss of generality.

Proposition 1. *Assume that the model is*

$$\begin{bmatrix}\Delta y_{f,t}\\ \Delta y_{d,t}\end{bmatrix} = \begin{bmatrix}\alpha_{11} & 0\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{bmatrix}\beta'_{11} & 0\\ \beta'_{12} & \beta'_{22}\end{bmatrix}\begin{bmatrix}y_{f,t-1}\\ y_{d,t-1}\end{bmatrix} + \sum_{i=1}^{p}\begin{bmatrix}\Gamma_{11i} & 0\\ \Gamma_{21i} & \Gamma_{22i}\end{bmatrix}\begin{bmatrix}\Delta y_{f,t-i}\\ \Delta y_{d,t-i}\end{bmatrix} + \varepsilon_t \quad (24)$$

where

1. *the error sequence* $\{\varepsilon_t\}$ *is such that: 1) it is a martingale difference sequence, 2) the strong law of large numbers applies so that* $T^{-1}\sum_{t=1}^{T} E(\varepsilon_t\varepsilon'_t|\mathcal{F}_{t-1}) \xrightarrow{a.s.} \Omega$ *where* $\mathcal{F}_{t-1}$ *is the filtration up to time* $t-1$*, 3) the fourth-order moments of* ε_t *are finite,*
2. α *and* β $(k \times r)$ *are full rank,*
3. $y_t \sim I(1)$ *so that* $\alpha'_\perp \Gamma \beta_\perp$ *is non-singular.*

Using (17)–(18) *for estimation, the asymptotic distribution of the long-run parameters is*

$$T\left(\begin{bmatrix}\mathrm{vec}(\hat{\beta}'_1)\\ \mathrm{vec}(\hat{\beta}'_{22})\end{bmatrix} - \begin{bmatrix}\mathrm{vec}(\beta'_1)\\ \mathrm{vec}(\beta'_{22})\end{bmatrix}\right) \xrightarrow{d} \left[\begin{array}{cc}\int G_{k,1}(u)G_{k,1}(u)'du \otimes \alpha'\Omega^{-1}\alpha & \int G_{k,1}(u)G_{k,2}(u)'du \otimes \alpha'\Omega^{-1}\alpha_{\cdot 2}\\ \int G_{k,2}(u)G_{k,1}(u)'du \otimes \alpha'_{\cdot 2}\Omega^{-1}\alpha & \int G_{k,2}(u)G_{k,2}(u)'du \otimes \alpha'_{\cdot 2}\Omega^{-1}\alpha_{\cdot 2}\end{array}\right]^{-1} \times \left[\begin{array}{c}\mathrm{vec}\left(\alpha'\Omega^{-1}\int dW_k G'_{k,1}\right)\\ \mathrm{vec}\left(\alpha'_{\cdot 2}\Omega^{-1}\int dW_k G'_{k,2}\right)\end{array}\right] \quad (25)$$

where $G_{k,1}(u)$ *and* $G_{k,2}(u)$ *denote the first* s *and last* $k-s$ *elements of* $G_k(u) = CW_k(u)$*, respectively; here,* $C = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp$*,* $\Gamma = I_k - \sum_{i=1}^{p-1}\Gamma_i$ *and* $W_k(u)$ *is a* k*-dimensional Brownian motion with covariance matrix* Ω*, where also* $\alpha'_{\cdot 2} = \begin{bmatrix}\alpha'_{12} & \alpha'_{22}\end{bmatrix}$*.*

Moreover, the asymptotic distribution of the adjustment and short-run parameters estimated based on (17)–(18) *is*

$$\sqrt{T}\left(\hat{\pi}_\alpha^{(GN)} - \pi_\alpha^{(GN)}\right) \xrightarrow{d} N_{r_1 s + r(k-s) + p(k-s)^2}(0, V) \quad (26)$$

where

$$V = \begin{bmatrix}\Sigma_{\beta\beta,11}\otimes(\Omega^{-1})_{11} & \Sigma_{\beta\beta,1\cdot}\otimes(\Omega^{-1})_{12} & \Sigma_{\beta 0,1\cdot}\otimes(\Omega^{-1})_{12}\\ \Sigma_{\beta\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{\beta\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{\beta 0}\otimes(\Omega^{-1})_{22}\\ \Sigma_{0\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{0\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{00}\otimes(\Omega^{-1})_{22}\end{bmatrix}^{-1}, \quad (27)$$

and $\Sigma_{\beta\beta} = \operatorname{plim} T^{-1}\sum_{t=1}^{T}\beta' y_t y'_t \beta$, $\Sigma_{\beta 0} = \operatorname{plim} T^{-1}\sum_{t=1}^{T}\beta' y_{t-1}\Delta y'_t$ *and* $\Sigma_{00} = \operatorname{plim} T^{-1}\sum_{t=1}\Delta y_t \Delta y'_t$*;* $\Sigma_{\beta\beta,11}$ *is the* $r_1 \times r_1$ *upper left block of* $\Sigma_{\beta\beta}$*, whereas* $\Sigma_{\beta\beta,\cdot 1}$ *and* $\Sigma_{\beta 0,\cdot 1}$ *refer to the first* r_1 *columns of the corresponding matrix. Similarly,* $(\Omega^{-1})_{ij}$ *is defined to be the* (i,j)*th block of* Ω^{-1} *(whose blocks have* s *or* $k-s$ *rows and/or columns).*

The proof is placed in the Appendix.

Remark 1. If the model is estimated under the weak exogeneity restriction $r_1 = 0$ and so

$$
\begin{aligned}
T \operatorname{vec}\left(\hat{\beta}^{(WE)'} - \beta^{(WE)'}\right) \\
&\xrightarrow{d} \left(\int G_k(u)G_k(u)'du \otimes \alpha'\Omega^{-1}\alpha\right)^{-1} \operatorname{vec}\left(\alpha'\Omega^{-1}\int dW_k G_k'\right) \\
&= \operatorname{vec}\left[\int dV_k G_k' \left(\int G_k(u)G_k(u)'du\right)^{-1}\right] \qquad (28)
\end{aligned}
$$

where $V_k(u) = (\alpha'\Omega^{-1}\alpha)^{-1}\alpha'\Omega^{-1}W_k(u)$. The latter form exactly mirrors the result by Johansen (1995).

Furthermore,

$$
\sqrt{T}\left(\hat{\pi}_\alpha^{(WE)} - \pi_\alpha^{(WE)}\right) \xrightarrow{d} N_{r(k-s)+p(k-s)^2}\left(0, \Sigma^{-1} \otimes (\Omega^{-1})_{22}^{-1}\right), \qquad (29)
$$

where

$$
\Sigma = \begin{bmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta 0} \\ \Sigma_{0\beta} & \Sigma_{00} \end{bmatrix} \qquad (30)
$$

Remark 2. If there are no foreign variables such that $r_1 = s = 0$, then

$$
\sqrt{T}\left(\hat{\pi}_\alpha - \pi_\alpha\right) \xrightarrow{d} N_{rk+pk^2}\left(0, \Sigma^{-1} \otimes \Omega\right) \qquad (31)
$$

and we obtain the same asymptotic distribution as derived by Lütkepohl (2005).

3 Monte Carlo Simulation

To analyze the small-sample properties we conduct a simulation study where we generate data according to (2). We use a five-variable system with three lags containing two foreign variables that are not cointegrated, and three domestic variables with two cointegrating relations. The sample sizes considered are $T = 100, 125, 150, \ldots, 500$. The parameter values are randomly chosen prior to the simulation and not altered. Equation (33) and Figure A1 in Appendix A display the model and its eigenvalues, respectively. The number of replicates is 5,000. Because one main purpose of many macroeconomic modeling exercises is to estimate impulse responses we evaluate the effect of small open economy restrictions using deviations from the true impulse responses in form of the mean squared error (MSE). The models we compare are the following VECMs: i) unrestricted, ii) restrictions on α (weak exogeneity), iii) restrictions on Γ_i (short-run restrictions), and iv) restrictions on both α and Γ_i (strong exogeneity). Because there are 25 impulse responses to present, we include a representative selection here and note that the remaining plots of mean squared errors are available upon request.

In Figures 1 and 2 we display the MSE for impulse responses as a function of the impulse response horizon and the sample size, respectively. The sample size in Figure 1 is 100 and the strongly exogenous VECM always has the lowest MSE while the unrestricted VECM has the largest. Occasionally, the weakly exogenous VECM is better than the VECM with short-run restrictions, as in the right two figures, and sometimes it is the other way around as in the top-left figure. The MSE always increases for the unrestricted VECM when increasing the impulse response horizon, while this is not always true for the other models. The interpretation of this phenomenon is that the restrictions are important for modeling the long-run relations.

Figure 2 shows the MSE as a function of sample size. As in the previous figure, the standard VECM has the largest MSE while the strongly exogenous VECM has the smallest. Also, as above, the ordering in terms of MSE between the VECM with only short-run dynamics versus only weakly exogenous restrictions is inconclusive, but always between the unrestricted VECM and the strongly exogenous model. As can be expected, as all models nest the true model, the MSE decreases with increased sample size. Sometimes the relative difference between sets of restrictions is small, as in the bottom-left figure, and in other cases the relative difference is larger, as in the top figures. Particularly for small sample sizes, the relative difference is often considerable. In summary, imposing the full set of restrictions when modeling small open economies will provide more accurate and precise estimates of the impulse responses.

4 Empirical Illustrations

4.1 A Strongly Exogenous Model for Sweden

The Swedish Ministry of Finance produces one of the most important GDP forecasts for the Swedish economy. As simple baseline models, the Ministry of Finance uses various types of vector autoregressive models, see e.g. Bjellerup and Shahnazarian (2012). We follow the same track and use a cointegrated VAR model, where the main variable of interest is the logarithm of GDP. Other variables in the model are a competitor-weighted exchange rate index, consumer price index, a foreign trade-weighted GDP (the US and the EU), interest on Swedish 3-month Treasury bills and unemployment. Similarly to Bjellerup and Shahnazarian (2012) we also have a dummy for the period 1991:Q3 to 1992:Q3. This dataset was previously used by Lyhagen et al. (2015), who investigated the effect of intercept correction on forecasts of GDP. Table 1 summarizes the data, which ranges from 1988:Q1 up to 2015:Q4.[1]

The specification of the VECM closely follows Lyhagen et al. (2015) with two cointegrating relations, found by using the p-values of MacKinnon et al. (1999), and four lags in levels. As a measure of evaluation we use the same as in the Monte Carlo simulation above, namely impulse responses. The size of the model is commonly found in the literature of empirical VEC models. The number of observations, $T = 116$, is typical for this type of application.

In Figure 3 examples of the impulse responses from the model of the Swedish economy are shown (the full set of impulse responses can be found in Figure B2 in Appendix B). As in the simulation in the previous section, we here consider one standard deviation shocks

[1]The data constitutes an extension of the dataset compared to Lyhagen et al. (2015), who used data for the period 1989:Q4–2012:Q2.

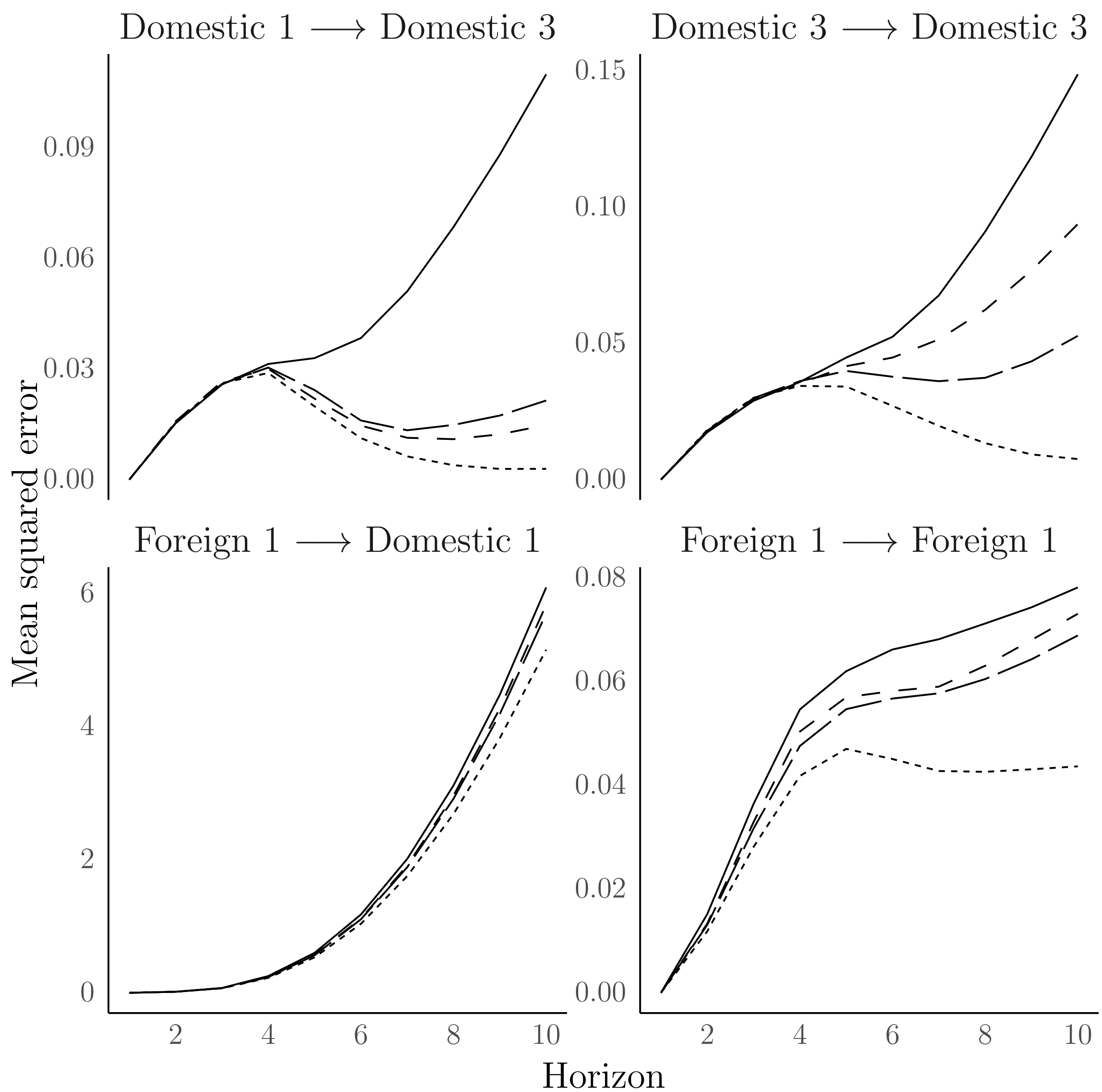

Figure 1: Mean squared error of impulse responses as a function of horizon, sample size $T = 100$. The four lines in the figures represent: unrestricted VECM (——), weakly exogenous VECM (— —), VECM with short-run restrictions (- - -) and strongly exogenous VECM (·········).

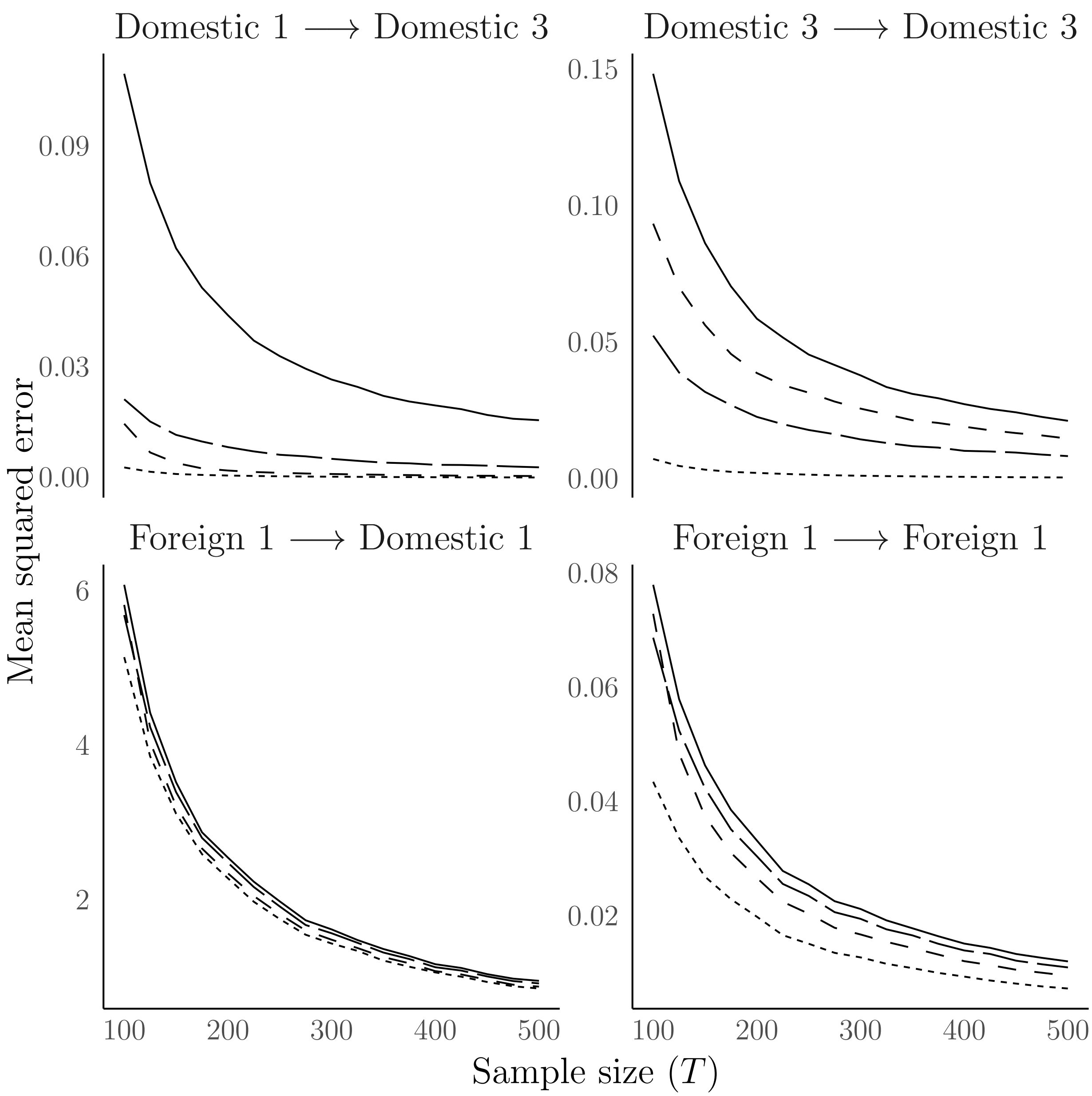

Figure 2: Mean squared error of impulse responses as a function of sample size at impulse response horizon $h = 10$. The four lines in the figures represent: unrestricted VECM (——), weakly exogenous VECM (– –), VECM with short-run restrictions (- - -) and strongly exogenous VECM (·········).

Table 1: Strongly exogenous model: Bjellerup and Shahnazarian (2012) data

Variable	*Description*
SWEGDP*	Seasonally adjusted real GDP, in logarithms
KIX	Competitor-weighted effective exchange rate index (log)
CPIX	Underlying inflation index (log)
TWGDP**	Foreign GDP as weighted between the US GDP and the Euro zone's GDP (log)
TB	Closing yield for a 3-months treasury bill
UNEMP	Relative unemployment
Dummy	Dummy variable for 1991:Q4–1992:Q3

Note: KIX, TB and UNEMP are aggregated to quarterly frequencies by taking averages of the corresponding months. The model includes a dummy variable for the period 1991:q3–1992:Q3.
*SWEGDP is sometimes referred to as GDP only for readability.
**The weights in TWGDP are 0.25 for the US and 0.75 for the Euro zone.

without any further identification scheme; the main purpose is to illustrate the differences obtained with changing sets of restrictions, and that message will remain when a proper identification approach is used. Overall, imposing both short-run dynamic restrictions and weak exogeneity sometimes leads to a different picture being painted than if none or only one type of restriction is enforced.

For example, in the top-right panel of Figure 3 the impulse responses of a one standard deviation shock of Swedish GDP on trade-weighted foreign GDP is displayed (SWEGDP $\rightarrow$ TWGDP). As TWGDP is assumed to be exogenous, the strongly exogenous VECM yields a straight line at zero (dotted line). Only restricting the short-run dynamics (dashed line) results in a negative impact of a shock to Swedish GDP on trade-weighted foreign GDP. When ignoring short-run restrictions a positive effect emerges with a larger impact of the unrestricted VECM (solid line) compared to the VECM with weak exogeneity (longdashed line). Thus, failing to cancel the channel from SWEGDP and TWGDP leads to the questionable result that Swedish GDP shocks affect trade-weighted GDP.

Switching to a one standard deviation shock of trade-weighted GDP and its effect on Swedish GDP in the bottom-right figure (TWGDP $\rightarrow$ SWEGDP) we find an initial positive impact for all models that eventually levels out at a positive long-term effect of around 1–1.5 and all models agree relatively well. In contrast, the effect of a one standard deviation shock of trade-weighted GDP on underlying inflation in the bottom-left figure (TWGDP $\rightarrow$ CPIX) yields a negative impulse response for the unrestricted VECM and the VECM with constrained short-run dynamics while there are positive effects indicated by the weakly and strongly exogenous models.

A shock to the three-month treasury bill yields either increases or decreases in inflation depending on the restrictions in the model (middle-right panel, TB $\rightarrow$ CPIX). Lastly, all models indicate an immediate decrease in unemployment when domestic GDP experiences a shock, but the trajectory going forward is notably more transitory for the models with restrictions as compared to the unrestricted model's response.

According to our empirical results there are sometimes large differences depending on the restrictions imposed. The results also clearly illustrate that occasionally models with weakly exogenous restrictions seem to behave similarly with or without restricted short-run dynamics, but that is not always the case. Hence, it is important to consider models enforcing restrictions on both the short-run adjustment parameter α as well as on the short-run dynamics Γ. While a model with no restrictions can still be consistently estimated under the setting in Section 2, by the nature of a small open economy the act of imposing such restrictions is uncontroversial for many applications.

4.2 A Granger Non-Causal Model for Sweden

Our second example originates from the work by Jacobson et al. (2001) who analyzed monetary policy and inflation in Sweden. Jacobson et al. (2001) imposed restrictions that are consistent with economic theory on the long-run relations (i.e. the cointegrating vectors) and demonstrated that these restrictions were useful both for policy analysis as well as for forecasting. We use the same dataset as Jacobson et al. (2001) and the variables are displayed in Table 2. Jacobson et al. (2001) argued that there should be four stochastic trends implying three cointegrating relations as there are seven variables in total. Three of the variables are foreign and four are domestic. The time period is 1972:Q2–1996:Q4 yielding a total of 99 observations. Additionally, five dummy variables for 'crashes' and 'changes in growth' capturing regime shifts in economic policy are used. The interpretation of the three cointegrating relations is that the first is a goods market equilibrium, the second is related to a financial markets equilibrium condition while the third consists of common trends and equilibrium conditions between the foreign variables. We use the same number of lags (four) and compare impulse responses from an unrestricted VECM, a VECM with Granger non-causality and a VECM with Granger non-causality and long-run restrictions.[2] All impulse responses can be found in Figure B3, while a selected subset are shown in Figure 4.

The response of domestic prices to a domestic interest rate shock (top-left panel, $i \to p$) is positive in all three models with similar paths along the impulse response horizon. Similarly, the response of the domestic interest rate to a foreign interest rate shock in the top-right panel ($i^f \to i$) is relatively similar across models with the same development, albeit at slightly different levels. The role of the exogeneity associated with a small open economy is

[2]The long-run restrictions follow Jacobson et al. (2001). Using the same ordering of variables as in Table 2, β is restricted to

$$\beta' = \begin{bmatrix} \beta_{11} & 1 & \beta_{12} & \beta_{13} & -1 & \beta_{14} & 1 \\ 0 & \beta_{21} & -1 & \beta_{22} & \beta_{23} & 1 & \beta_{24} \\ \beta_{31} & \beta_{32} & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

The preceding restrictions are overidentifying, and Jacobson et al. (2001) estimate the model using both a different set of overidentifying restrictions involving more constraints, and a relaxed set of restrictions that make the above restrictions exactly identifying. Because we compute impulse responses based on $\alpha\beta'$, exact identification of β still yields the same impulse responses, and so our unrestricted model corresponds to the model with exact identification used by Jacobson et al. (2001). We stress, however, that we do not consider any structural identification of shocks, but rather investigate the role of small open economy restrictions on impulse responses for which a reduced form analysis suffices.

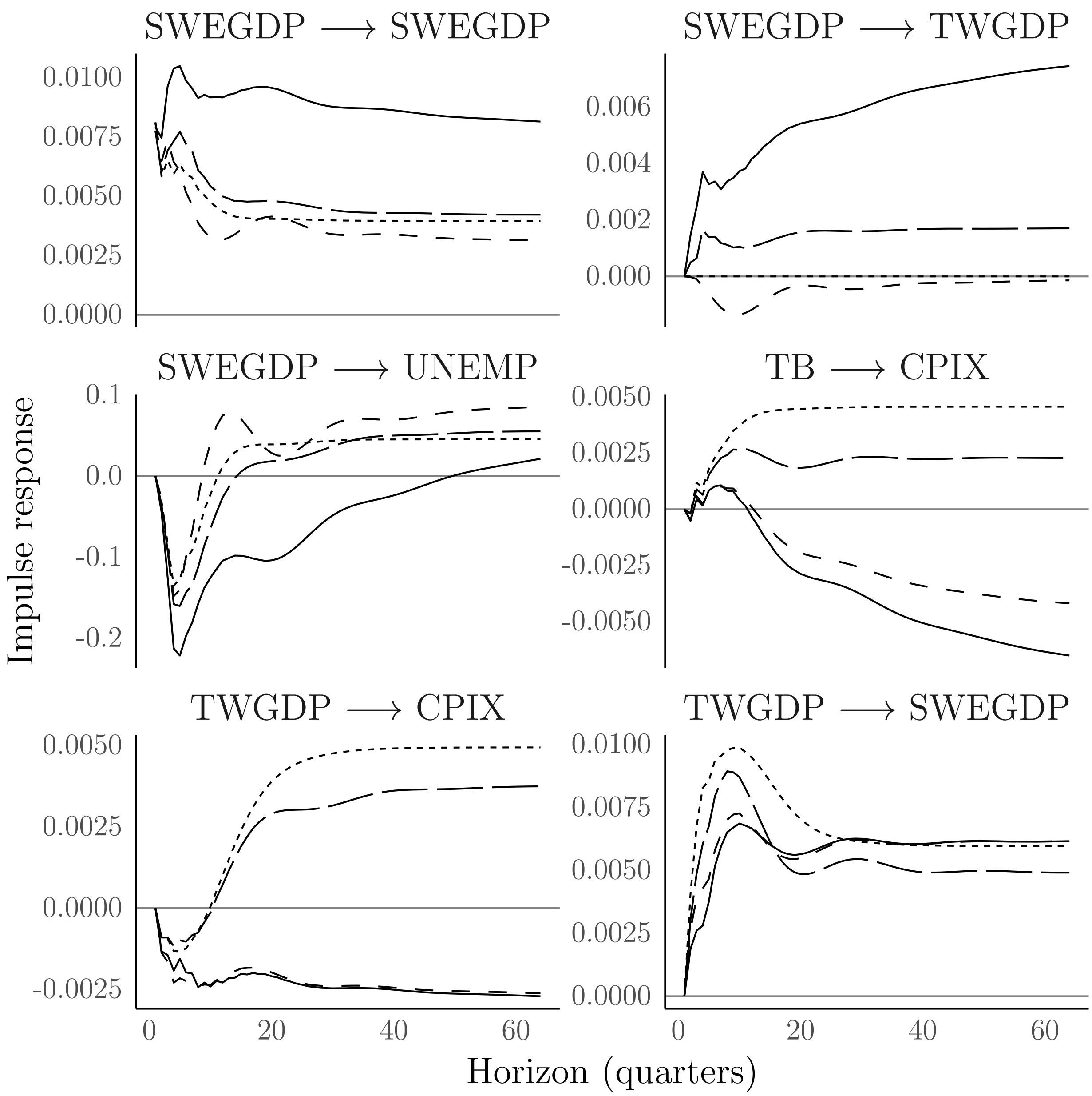

Figure 3: Examples of impulse responses to 1 SD shocks for the model of the Swedish economy (Section 4.1). The four lines in the figures represent: unrestricted VECM (——), weakly exogenous VECM (— —), VECM with short-run restrictions (- - -) and strongly exogenous VECM (·········).

Table 2: Granger non-causal model: Jacobson et al. (2001) data

Variable	*Description*
y^f	Foreign real output, $y_t^f = 100\ln(Y_t^f)$ where Y_t^f is German real GDP in 1991 prices
p^f	Foreign price levels, $p_t^f = 100\ln(P_t^f)$, where P_t^f is the geometric sum of Sweden's 20 most important trading partners weighted by IMF's TCW index
i^f	Foreign nominal interest rate, $i_t^f = 100\ln(1 + I_t^f/100)$ where I_t^f is the German three-month treasury bills rate
y	Swedish real output, $y_t = 100\ln(Y_t)$ where Y_t is Swedish real GDP in 1991 prices
p	Swedish price levels, $p_t = 100\ln(P_t)$, where P_t is the quarterly average of Swedish CPI
i	Swedish nominal interest rate, $i_t = 100\ln(1 + I_t/100)$ where I_t is the Swedish three-month treasury bills rate
e	Nominal exchange rate, $e_t = 100\ln(S_t)$ where S_t is the geometric sum of the nominal Krona exchange rate of Sweden's top 20 trading partners using the TCW index

Note: The model also includes five dummy variables, see Jacobson et al. (2001) for details.

clearly visible in the middle-left panel displaying the effect of a domestic price level shock on foreign interest rates; the unrestricted VAR estimates that the foreign interest rate reacts positively, whereas the two models with Granger non-causality restrict the response to be zero. For the response of the domestic price level to a foreign price level shock (middle-right panel, $p^f \to p$), the models exhibit quite different behaviors. The unrestricted VECM shows a drastic increase in the domestic price level, whereas the Granger non-causal model with long-run restrictions quickly stabilizes at a much lower (but positive) level. The Granger non-causal model without any long-run restrictions instead shows a positive response in the initial periods followed by a slow drift towards its negative asymptote. The bottom-right panel showing the response of the price level to a real output shock ($y \to p$) reveals again a more sizable difference between the Granger non-causal models and the unrestricted model, where the latter indicates a much larger response of prices. Finally, the bottom-right panel displays how foreign price levels respond to a foreign real output shock. The Granger non-causal models present positive responses, whereas the unrestricted VAR estimates a negative response.

In summary, the above presentation of the results makes it clear that small open economy restrictions can be important for impulse response analysis. The benefit of such restrictions is that they increase the efficiency of estimation, which may be important due to the small sample sizes common in applications, and are typically uncontroversial as the property of a small open economy is widely accepted for many countries.

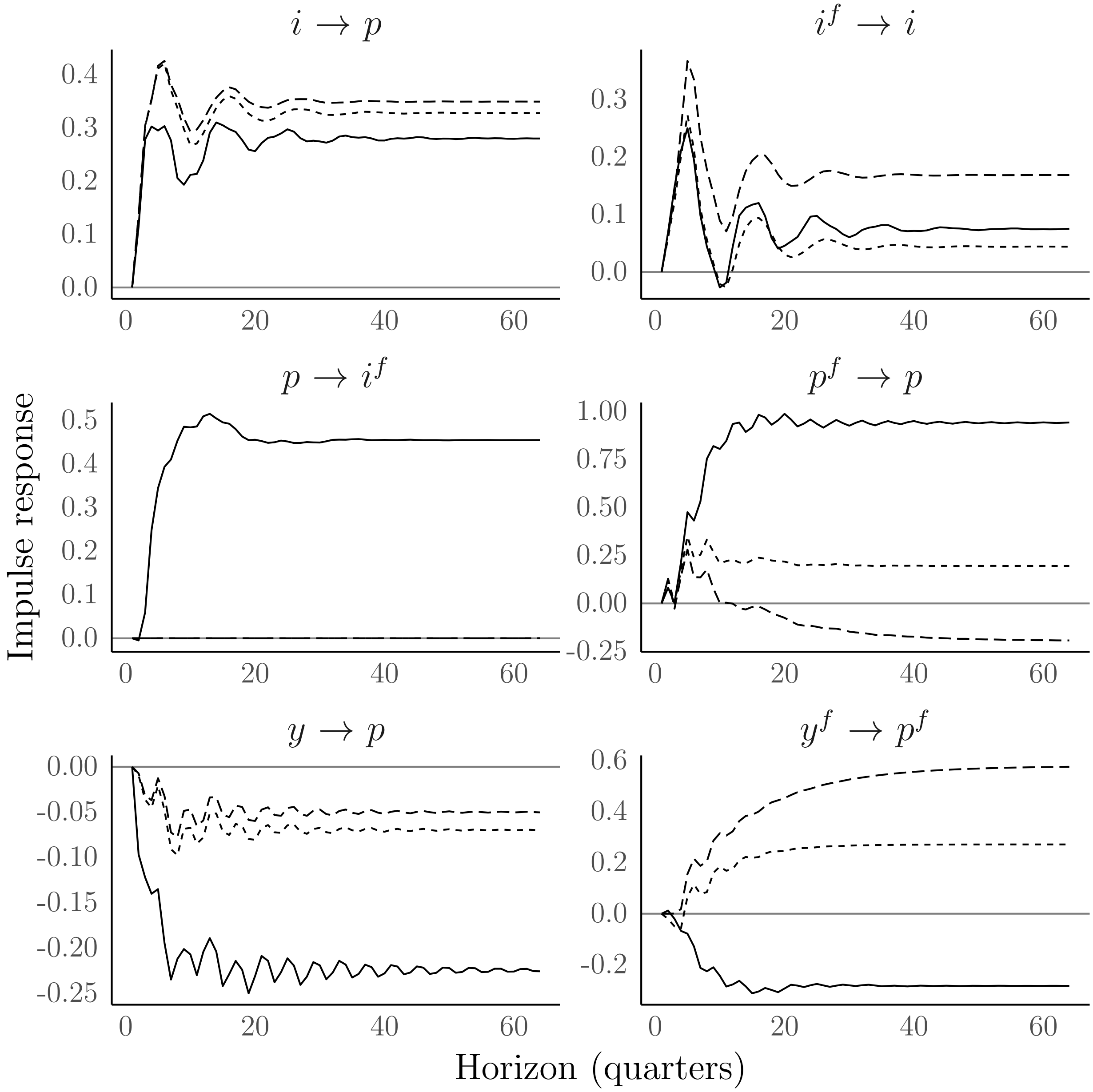

Figure 4: Impulse responses to 1 SD shocks for the Granger non-causal model of the Swedish economy (Section 4.2). The three lines in the figures represent standard VECM (——), VECM with Granger non-causality (– –) and VECM with Granger non-causality and long-run restrictions (- - -).

5 Conclusions

In this paper we have proposed an estimation procedure in the case of exogeneity restrictions in vector error correction models (VECM). We have focused on the perspective of modeling small open economies as doing so naturally imposes restrictions on the VECM. A Monte Carlo simulation study is conducted in order to show the advantages of imposing such restrictions. We find that it is beneficial with respect to the estimation of impulse responses to impose the full suite of restrictions in the model when the underlying model is in accordance with the small open economy property. Ignoring restrictions will most often substantially increase the MSE. Using one set of restrictions is typically notably better than no restrictions, but worse than using both. We apply our method to two Swedish macroeconomic datasets and estimate one VECM with a strongly exogenous foreign variable, and one VECM with a foreign block of variables for which the domestic variables are Granger non-causal. The empirical examples demonstrate that vastly different results betweenbmatrix the models with or without restrictions can be obtained. The advantage of the small open economy restrictions is that they are often uncontroversial, and our approach thus offers a useful method for incorporating undebatable economic theory into the statistical estimation of the model.

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*Appendix

A Simulation Details

The parameters of the data-generating process are presented in (33) and the roots of the model are displayed in Figure A1.

$$
\begin{aligned}
\alpha &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.2785 & 0.5469 \\ -0.5 & 0.25 \\ 0.25 & -0.5 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0.8147 & 0.9134 \\ 0.9058 & 0.6324 \\ 0.127 & 0.0975 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\Omega &= \begin{bmatrix} 1 & 0.25 & 0.0625 & 0.0156 & 0.0039 \\ 0.25 & 1 & 0.25 & 0.0625 & 0.0156 \\ 0.0625 & 0.25 & 1 & 0.25 & 0.0625 \\ 0.0156 & 0.0625 & 0.25 & 1 & 0.25 \\ 0.0039 & 0.0156 & 0.0625 & 0.25 & 1 \end{bmatrix} \\
\Gamma_1 &= \begin{bmatrix} 0.2288 & -0.1712 & 0 & 0 & 0 \\ 0.2324 & 0.2353 & 0 & 0 & 0 \\ 0.2286 & -0.1791 & 0.1461 & -0.2321 & 0.0894 \\ -0.0073 & -0.0391 & 0.2297 & 0.1746 & 0.1289 \\ 0.1501 & 0.2079 & 0.0779 & 0.217 & 0.1216 \end{bmatrix}, \\
\Gamma_2 &= \begin{bmatrix} -0.0269 & -0.0822 & 0 & 0 & 0 \\ 0.0389 & 0.0515 & 0 & 0 & 0 \\ -0.117 & -0.1007 & -0.0457 & -0.0153 & 0.0738 \\ -0.0558 & 0.0809 & 0.1126 & -0.0296 & -0.0783 \\ -0.1135 & 0.0487 & -0.1164 & 0.0664 & -0.0026 \end{bmatrix}, \\
\Gamma_3 &= \begin{bmatrix} -0.0136 & 0.0523 & 0 & 0 & 0 \\ 0.0366 & 0.0637 & 0 & 0 & 0 \\ -0.056 & -0.0843 & 0.1149 & -0.069 & 0.0015 \\ 0.0449 & -0.0953 & -0.0399 & 0.0628 & 0.0498 \\ 0.0388 & -4e-04 & 0.0213 & -0.0612 & 0.0977 \end{bmatrix}.
\end{aligned}
\tag{33}
$$

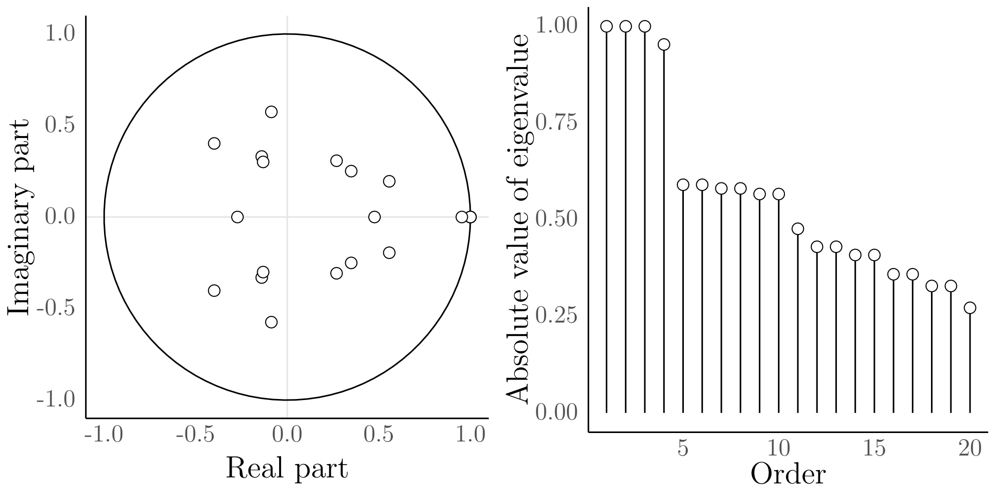

Figure A1: Characteristics of the simulation study's data-generating process. Left: inverse roots of AR characteristic polynomial. Right: absolute values of eigenvalues in decreasing order. As noted in the text, there are two cointegrating relations and thus three unit roots.

B Impulse Responses

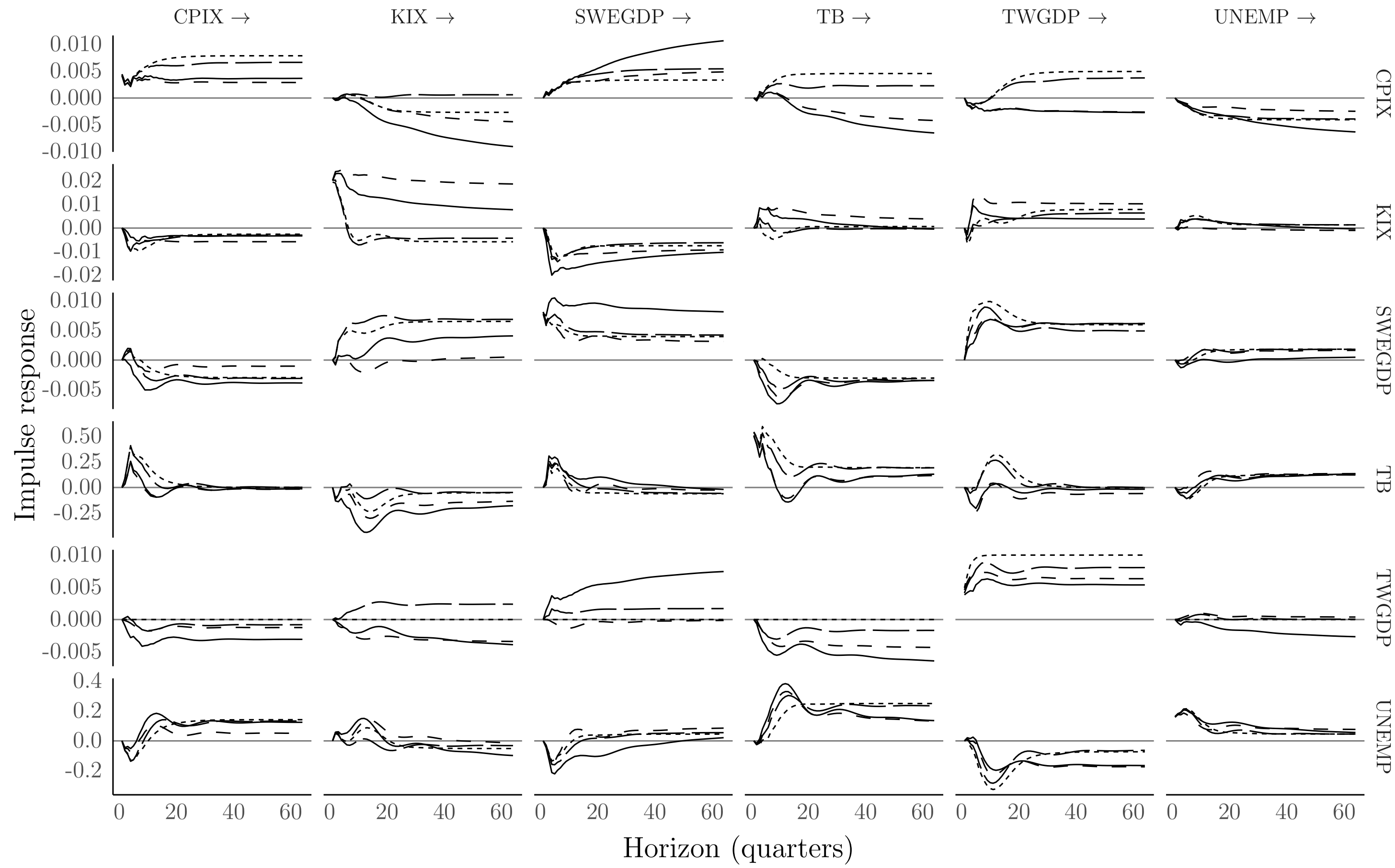

Figure B2: Impulse responses to 1 SD shocks for the weakly exogenous model of the Swedish economy (Section 4.1). Columns denote the origin of the shock and rows the response. The four lines in the figures represent standard VECM (——), VECM with weak exogeneity (– –), VECM with short-run restrictions (- - -) and VECM with short-run restrictions and weak exogeneity (⋯⋯).

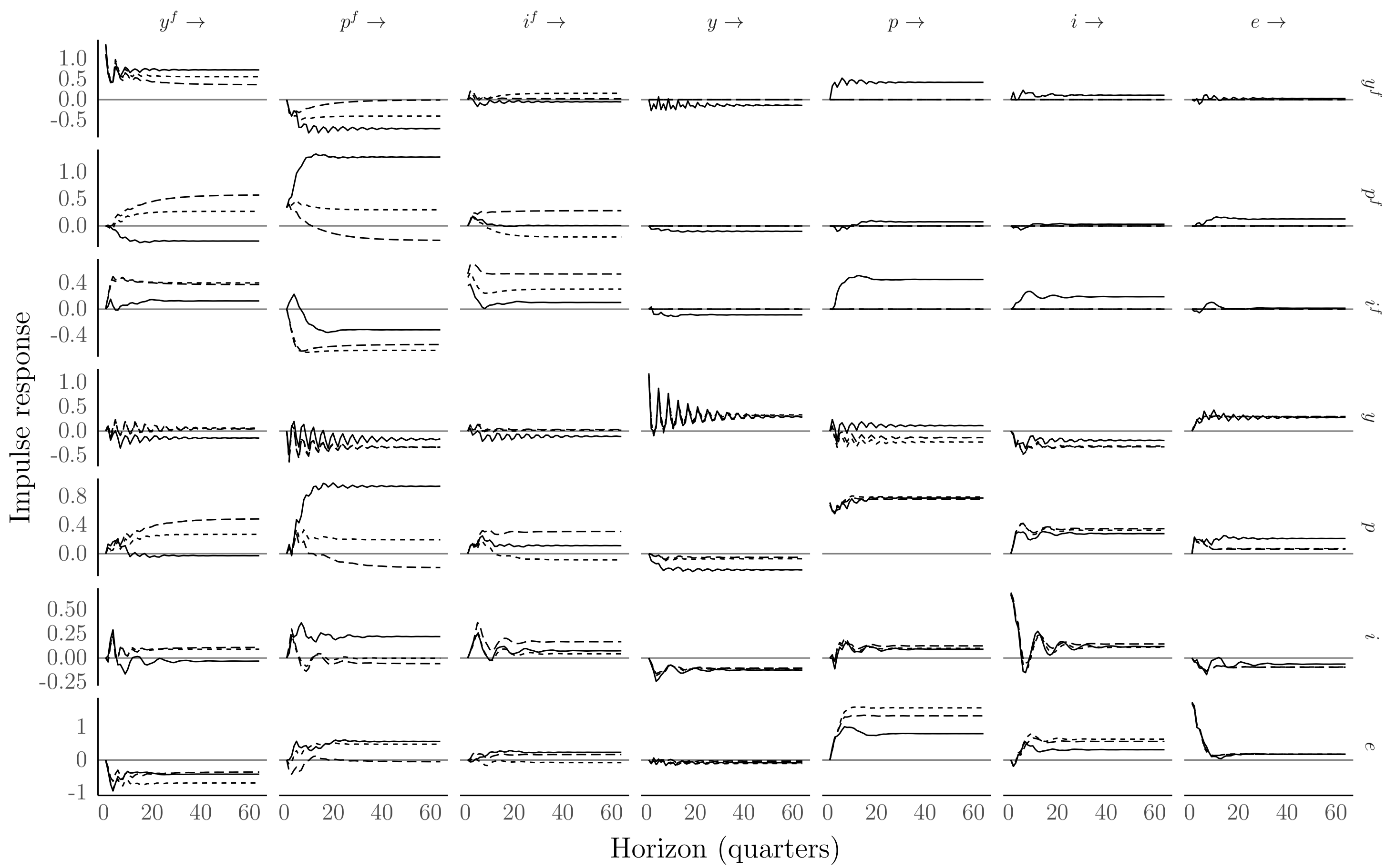

Figure B3: Impulse responses to 1 SD shocks for the Granger non-causal model of the Swedish economy (Section 4.2). Columns denote the origin of the shock and rows the response. The three lines in the figures represent standard VECM (——), VECM with Granger non-causality (– –) and VECM with Granger non-causality and long-run restrictions (- - -).

C Proof of Proposition 1

Following Magnus (1978); Groen and Kleibergen (2003) the asymptotic distributions of the seemingly unrelated regressions-type estimators used are the same after one iteration as after full convergence. Moreover, because the model parameters can be consistently estimated in the full model and the iterative procedure produces non-decreasing steps on the log-likelihood surface (Boswijk, 1995; Groen and Kleibergen, 2003; Boswijk and Doornik, 2004) the iterative procedure is also consistent.

To derive the asymptotic distributions, we first introduce a lemma with standard results.

Lemma 1. *Let $W_k(u) = \Omega^{1/2}B_k(u)$ where $B_k(u)$ is a standard k-dimensional Brownian motion. Furthermore, let $C = \beta_\perp(\alpha_\perp'\Gamma\beta_\perp)\alpha_\perp'$ where $\Gamma = I - \sum_{i=1}^{p-1}\Gamma_i$ and let $G_k(u) = CW_k(u)$. Moreover, define $\ddot{\Upsilon}_T = \operatorname{diag}(T^{-1}I_k, T^{-1/2}I_{p(k-s)})$. Then*

$$T^{-1/2}\beta_\perp' y_{[Tu]} \xrightarrow{d} CW_k(u) \tag{34}$$

$$T^{-1/2}\Delta y_{[Tu]} \xrightarrow{p} 0 \tag{35}$$

$$\ddot{\Upsilon}_T\tilde{Y}_{-1}'\tilde{Y}_{-1}\ddot{\Upsilon}_T \xrightarrow{p} \begin{bmatrix} \int G_k(u)G_k(u)'du & 0 \\ 0 & \Sigma_{00} \end{bmatrix} \tag{36}$$

$$\ddot{\Upsilon}_T\tilde{Y}_{-1}'\varepsilon \xrightarrow{d} \begin{bmatrix} \int G_k dW_k' \\ \xi \end{bmatrix} \tag{37}$$

$$T^{-1}\beta'\tilde{y}_{-1}'\tilde{y}_{-1}\beta \xrightarrow{p} \Sigma_{\beta\beta} \tag{38}$$

$$T^{-1}\beta'\tilde{y}_{-1}'\Delta\tilde{Y} \xrightarrow{p} \Sigma_{\beta 0} \tag{39}$$

$$T^{-1}\Delta\tilde{Y}'\Delta\tilde{Y} \xrightarrow{p} \Sigma_{00} \tag{40}$$

$$T^{-1/2}\begin{bmatrix} \beta'\tilde{y}_{-1}'\tilde{\varepsilon} \\ \Delta Y'\tilde{\varepsilon} \end{bmatrix} \xrightarrow{d} \begin{bmatrix} \zeta \\ \xi \end{bmatrix} \tag{41}$$

where

$$\operatorname{vec}\begin{bmatrix} \zeta \\ \xi \end{bmatrix} \sim N_{k(r+pk)}(0, \Omega\otimes\Sigma) \tag{42}$$

$$\Sigma = \begin{bmatrix} \Sigma_{\beta\beta} & \Sigma_{\beta 0} \\ \Sigma_{0\beta} & \Sigma_{00} \end{bmatrix}. \tag{43}$$

Proof. These results are standard in the literature and follow from Johansen (1995, Lemma 10.2–10.3) and Hamilton (1994, Proposition 18.1). □

Before deriving the asymptotic distributions, a note on notation is in order. We let $(\Omega^{-1})_{ij}$ denote the (i,j)th block of Ω^{-1} where $i,j = 1,2$. Likewise, $(\Omega^{-1})_{i\cdot}$ refers to the ith block of rows across all columns, i.e. $\begin{bmatrix}(\Omega^{-1})_{i1} & (\Omega^{-1})_{i2}\end{bmatrix}$, and vice versa for $(\Omega^{-1})_{\cdot j}$. A similar notation is also used for α, where $\alpha_{\cdot 2}$ refers to the second block of columns (across all rows), as well as for $\Sigma_{\beta\beta}$, where $\Sigma_{\beta\beta,11}$ is the top-left block and $\Sigma_{\beta\beta,1\cdot}$ the upper block of $\Sigma_{\beta\beta}$.

C.1 Asymptotic Distribution of $\hat{\pi}_\alpha$ Under Granger Non-Causality

Let

$$\alpha = \left[\begin{array}{c|c} \alpha_{11} & \alpha_{12} = 0_{s\times r_2} \\ \hline \alpha_{21} & \alpha_{22} \end{array}\right] \tag{44}$$

where α_{11} is $s\times r_1$, α_{21} is $(k-s)\times r_1$ and α_{22} is $(k-s)\times r_2$, where r_1 denotes cointegrating relations that enter the foreign equations. For convenience, let also $\alpha_1 = (\alpha_{11}, \alpha_{12})$, $\alpha_2 = (\alpha_{21}, \alpha_{22})$ and $\Gamma_d = \begin{bmatrix}\Gamma_{221} & \cdots & \Gamma_{22p}\end{bmatrix}$.

The zero restrictions can be imposed by the following decomposition

$$\mathrm{vec}(\Pi') = \begin{bmatrix}\mathrm{vec}(\alpha\beta') \\ \mathrm{vec}\begin{bmatrix}0 \\ \Gamma_d\end{bmatrix}\end{bmatrix} = \underbrace{\begin{bmatrix}(\beta\otimes I_k) & 0_{rk\times p(k-s)^2} \\ 0_{pk(k-s)\times rk} & I_{p(k-s)}\otimes\begin{bmatrix}0_{s\times k-s} \\ I_{k-s}\end{bmatrix}\end{bmatrix}}_{F_{1,\beta}(\beta)} \tag{45}$$

$$\times \underbrace{\begin{bmatrix} K_{r,k}\begin{bmatrix} I_s\otimes\begin{bmatrix}I_{r_1} \\ 0_{r_2\times r_1}\end{bmatrix} \\ 0_{r(k-s)\times r_1 s}\end{bmatrix} K_{s,r_1} & I_r\otimes\begin{bmatrix}0_{s\times k-s} \\ I_{k-s}\end{bmatrix} & 0_{rk\times p(k-s)^2} \\ 0_{p(k-s)^2\times r_1 s} & 0_{p(k-s)^2\times r(k-s)} & I_{p(k-s)^2}\end{bmatrix}}_{F_{2,\beta}} \begin{bmatrix}\mathrm{vec}(\alpha_{11}) \\ \mathrm{vec}(\alpha_2) \\ \mathrm{vec}(\Gamma_d)\end{bmatrix}. \tag{46}$$

Let $F_{0,\beta}(\beta) = F_{1,\beta}(\beta)F_{2,\beta}$. Using that

$$\tilde{Y}'_{-1}\otimes I_k = \begin{bmatrix}\tilde{y}'_{-1}\otimes I_k \\ \Delta\tilde{Y}'\otimes I_k\end{bmatrix} \tag{47}$$

and

$$F_{1,\beta}(\beta)'(\tilde{Y}'_{-1}\otimes I_k)K_{T,k} = \begin{bmatrix}\beta'\tilde{y}'_{-1}\otimes I_k \\ \Delta Y'\otimes(0_{k-s\times s}, I_{k-s})\end{bmatrix} K_{T,k} \tag{48}$$

gives

$$F_{0,\beta}(\beta)'(I_k\otimes\tilde{Y}'_{-1}) = F'_{2,\beta}F_{1,\beta}(\beta)'(\tilde{Y}'_{-1}\otimes I_k)K_{T,k} \tag{49}$$

$$= F'_{2,\beta}\begin{bmatrix}\beta'\tilde{y}'_{-1}\otimes I_k \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} K_{T,k} \tag{50}$$

$$= \begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix} & 0_{r_1 s\times r(k-s)}\end{bmatrix} K_{k,r} & 0_{r_1 s\times p(k-s)^2} \\ I_r\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} & 0_{r(k-s)\times p(k-s)^2} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)^2}\end{bmatrix} \tag{51}$$

$$\times\begin{bmatrix}\beta'\tilde{y}'_{-1}\otimes I_k \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} K_{T,k} \tag{52}$$

$$= \begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix} & 0_{r_1 s\times r(k-s)}\end{bmatrix} K_{k,r}(\beta'\tilde{y}'_{-1}\otimes I_k) \\ \left(I_r\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\right)\otimes(\beta'\tilde{y}'_{-1}\otimes I_k) \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} K_{T,k} \tag{53}$$

$$= \begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix} & 0_{r_1 s\times r(k-s)}\end{bmatrix}\begin{bmatrix}I_s\otimes\beta'\tilde{y}'_{-1} & 0 \\ 0 & I_{k-s}\otimes\beta'\tilde{y}'_{-1}\end{bmatrix} K_{k,T} \\ \beta'\tilde{y}'_{-1}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} K_{T,k} \tag{54}$$

$$= \begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\beta'_1\tilde{y}'_{-1} & 0_{r_1 s\times T(k-s)}\end{bmatrix} K_{k,T} \\ \beta'\tilde{y}'_{-1}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} K_{T,k} \tag{55}$$

and it follows that

$$T^{-1}F_{0,\beta}(\beta)'\left[\Omega^{-1}\otimes\left(\tilde{Y}'_{-1}\tilde{Y}_{-1}\right)\right]F_{0,\beta}(\beta) \tag{56}$$

$$= T^{-1}F'_{2,\beta}F_{1,\beta}(\beta)'(\tilde{Y}'_{-1}\otimes I_k)K_{T,k}\left(\Omega^{-1}\otimes I_T\right)K_{k,T}(\tilde{Y}_{-1}\otimes I_k)F_{1,\beta}(\beta)F_{2,\beta} \tag{57}$$

$$= T^{-1}\begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\beta'_1\tilde{y}'_{-1} & 0_{r_1 s\times T(k-s)}\end{bmatrix} K_{k,T} \\ \beta'\tilde{y}'_{-1}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix}(I_T\otimes\Omega^{-1}) \tag{58}$$

$$\times\begin{bmatrix} K_{r_1,s}\begin{bmatrix}I_s\otimes\beta'_1\tilde{y}'_{-1} & 0_{r_1 s\times T(k-s)}\end{bmatrix} K_{k,T} \\ \beta'\tilde{y}'_{-1}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} \\ \Delta Y'\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix}'. \tag{59}$$

The resulting matrix is a 3×3 symmetric block matrix. The blocks are in turn:

$$
\begin{aligned}
(1,1) &= K_{r_1,s}\begin{bmatrix} I_s \otimes \beta_1' \tilde{y}_{-1}' & 0_{r_1 s \times T(k-s)} \end{bmatrix} K_{k,T}(I_T \otimes \Omega^{-1}) K_{T,k} \begin{bmatrix} I_s \otimes \tilde{y}_{-1}\beta_1 \\ 0_{T(k-s)\times r_1 s} \end{bmatrix} K_{s,r_1} \\
&= K_{r_1,s}\begin{bmatrix} I_s \otimes \beta_1' \tilde{y}_{-1}' & 0_{r_1 s \times T(k-s)} \end{bmatrix} \begin{bmatrix} (\Omega^{-1})_{11} \otimes I_T & (\Omega^{-1})_{12} \otimes I_T \\ (\Omega^{-1})_{21} \otimes I_T & (\Omega^{-1})_{22} \otimes I_T \end{bmatrix} \\
&\quad \times \begin{bmatrix} I_s \otimes \tilde{y}_{-1}\beta_1 \\ 0_{T(k-s)\times r_1 s} \end{bmatrix} K_{s,r_1} \\
&= K_{r_1,s}(I_s \otimes \beta_1' \tilde{y}_{-1}')((\Omega^{-1})_{11} \otimes I_T)(I_s \otimes \tilde{y}_{-1}\beta_1) K_{s,r_1} \\
&= \beta_1' \tilde{y}_{-1}' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{11} \\
(2,1) &= \left(\beta' \tilde{y}_{-1}' \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix}\right)(I_T \otimes \Omega^{-1}) K_{T,k} \begin{bmatrix} I_s \otimes \tilde{y}_{-1}\beta_1 \\ 0_{T(k-s)\times r_1 s} \end{bmatrix} K_{s,r_1} \\
&= K_{r,k-s}\left(\begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \otimes \beta' \tilde{y}_{-1}'\right) \begin{bmatrix} (\Omega^{-1})_{11} \otimes \tilde{y}_{-1}\beta_1 \\ (\Omega^{-1})_{21} \otimes \tilde{y}_{-1}\beta_1 \end{bmatrix} K_{s,r_1} \\
&= \beta' \tilde{y}_{-1}' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{21} \\
(3,1) &= \left(\Delta Y' \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix}\right)(I_T \otimes \Omega^{-1}) K_{T,k} \begin{bmatrix} I_s \otimes \tilde{y}_{-1}\beta_1 \\ 0_{T(k-s)\times r_1 s} \end{bmatrix} K_{s,r_1} \\
&= K_{r,k-s}\left(\begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \otimes \Delta Y'\right) \begin{bmatrix} (\Omega^{-1})_{11} \otimes \tilde{y}_{-1}\beta_1 \\ (\Omega^{-1})_{21} \otimes \tilde{y}_{-1}\beta_1 \end{bmatrix} K_{s,r_1} \\
&= \Delta Y' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{21} \\
(2,2) &= \left(\beta' \tilde{y}_{-1}' \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix}\right)(I_T \otimes \Omega^{-1}) \left(\tilde{y}_{-1}\beta \otimes \begin{bmatrix} 0_{s\times k-s} \\ I_{k-s} \end{bmatrix}\right) \\
&= \beta' \tilde{y}_{-1}' \tilde{y}_{-1}\beta \otimes (\Omega^{-1})_{22} \\
(3,2) &= \left(\Delta Y' \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix}\right)(I_T \otimes \Omega^{-1}) \left(\tilde{y}_{-1}\beta \otimes \begin{bmatrix} 0_{s\times k-s} \\ I_{k-s} \end{bmatrix}\right) \\
&= \Delta Y' \tilde{y}_{-1}\beta \otimes (\Omega^{-1})_{22} \\
(3,3) &= \left(\Delta Y' \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix}\right)(I_T \otimes \Omega^{-1}) \left(\Delta Y \otimes \begin{bmatrix} 0_{s\times k-s} \\ I_{k-s} \end{bmatrix}\right) \\
&= \Delta Y' \Delta Y \otimes (\Omega^{-1})_{22}
\end{aligned} \tag{60}
$$

resulting in

$$
T^{-1} F_{0,\beta}(\beta)\left[\Omega^{-1} \otimes \left(\tilde{Y}_{-1}' \tilde{Y}_{-1}\right)\right] F_{0,\beta}(\beta) \tag{61}
$$

$$
= T^{-1}\begin{bmatrix} \beta_1' \tilde{y}_{-1}' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{11} & & \\ \beta' \tilde{y}_{-1}' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{21} & \beta' \tilde{y}_{-1}' \tilde{y}_{-1}\beta \otimes (\Omega^{-1})_{22} & \\ \Delta Y' \tilde{y}_{-1}\beta_1 \otimes (\Omega^{-1})_{21} & \Delta Y' \tilde{y}_{-1}\beta \otimes (\Omega^{-1})_{22} & \Delta Y' \Delta Y \otimes (\Omega^{-1})_{22} \end{bmatrix}. \tag{62}
$$

By Lemma 1

$$
T^{-1} F_{0,\beta}(\beta)\left[\Omega^{-1} \otimes \left(\tilde{Y}_{-1}' \tilde{Y}_{-1}\right)\right] F_{0,\beta}(\beta) \xrightarrow{p} \tag{63}
$$

$$
\begin{bmatrix} \Sigma_{\beta\beta,11} \otimes (\Omega^{-1})_{11} & \Sigma_{\beta\beta,1\cdot} \otimes (\Omega^{-1})_{12} & \Sigma_{\beta 0,1\cdot} \otimes (\Omega^{-1})_{12} \\ \Sigma_{\beta\beta,\cdot 1} \otimes (\Omega^{-1})_{21} & \Sigma_{\beta\beta} \otimes (\Omega^{-1})_{22} & \Sigma_{\beta 0} \otimes (\Omega^{-1})_{22} \\ \Sigma_{0\beta,\cdot 1} \otimes (\Omega^{-1})_{21} & \Sigma_{0\beta} \otimes (\Omega^{-1})_{22} & \Sigma_{00} \otimes (\Omega^{-1})_{22} \end{bmatrix}. \tag{64}
$$

Furthermore,

$$F_{1,\beta}(\beta)' K_{k+p(k-s),k} \left(\Omega^{-1} \otimes \tilde{Y}_{-1}' \right) \operatorname{vec}(\tilde{\varepsilon}) \tag{65}$$

$$= F_{1,\beta}(\beta)' \left(\tilde{Y}_{-1}' \otimes \Omega^{-1} \right) K_{T,k} \operatorname{vec}(\tilde{\varepsilon}) \tag{66}$$

$$= \begin{bmatrix} \beta' \tilde{y}_{-1}' \otimes \Omega^{-1} \\ \left(I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \right) \left(\Delta Y' \otimes \Omega^{-1} \right) \end{bmatrix} K_{T,k} \operatorname{vec}(\tilde{\varepsilon}) \tag{67}$$

$$= \begin{bmatrix} I_{rk} & 0 \\ 0 & I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \end{bmatrix} \left(\begin{bmatrix} \beta' \tilde{y}_{-1}' \\ \Delta Y' \end{bmatrix} \otimes \Omega^{-1} \right) K_{T,k} \operatorname{vec}(\tilde{\varepsilon}) \tag{68}$$

$$= \begin{bmatrix} I_{rk} & 0 \\ 0 & I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \end{bmatrix} K_{r+p(k-s),k} \left(\Omega^{-1} \otimes \begin{bmatrix} \beta' \tilde{y}_{-1}' \\ \Delta Y' \end{bmatrix} \right) \operatorname{vec}(\tilde{\varepsilon}) \tag{69}$$

$$= \begin{bmatrix} I_{rk} & 0 \\ 0 & I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \end{bmatrix} K_{r+p(k-s),k} \left(\Omega^{-1} \otimes I_{r+p(k-s)} \right) \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{70}$$

$$= \begin{bmatrix} I_{rk} & 0 \\ 0 & I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \end{bmatrix} \left(I_{r+p(k-s)} \otimes \Omega^{-1} \right) K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{71}$$

$$= \begin{bmatrix} I_r \otimes \Omega^{-1} & 0 \\ 0 & \left(I_{p(k-s)} \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} \right) \left(I_{p(k-s)} \otimes \Omega^{-1} \right) \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{72}$$

$$= \begin{bmatrix} I_r \otimes \Omega^{-1} & 0 \\ 0 & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{73}$$

and so

$$F_{0,\beta}(\beta)' \left(\Omega^{-1} \otimes I_{k+p(k-s)} \right) \operatorname{vec} \left(\tilde{Y}_{-1}' \tilde{\varepsilon} \right) \tag{74}$$

$$= F_{2,\beta}' F_{1,\beta}(\beta)' K_{k+p(k-s),k} \left(\Omega^{-1} \otimes \tilde{Y}_{-1}' \right) \operatorname{vec}(\tilde{\varepsilon}) \tag{75}$$

$$= F_{2,\beta}' \begin{bmatrix} I_r \otimes \Omega^{-1} & 0_{rk\times pk(k-s)} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{76}$$

$$= \begin{bmatrix} K_{r_1,s} \begin{bmatrix} I_s \otimes \begin{bmatrix} I_{r_1} & 0_{r_1\times r_2} \end{bmatrix} & 0_{r_1 s\times r(k-s)} \end{bmatrix} K_{k,r} & 0_{r_1 s\times p(k-s)^2} \\ I_r \otimes \begin{bmatrix} 0_{k-s\times s} & I_{k-s} \end{bmatrix} & 0_{r(k-s)\times p(k-s)^2} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)^2} \end{bmatrix} \tag{77}$$

$$\times \begin{bmatrix} I_r \otimes \Omega^{-1} & 0_{rk\times pk(k-s)} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{78}$$

$$= \begin{bmatrix} K_{r_1,s} \begin{bmatrix} I_s \otimes \begin{bmatrix} I_{r_1} & 0_{r_1\times r_2} \end{bmatrix} & 0_{r_1 s\times r(k-s)} \end{bmatrix} (\Omega^{-1} \otimes I_r) K_{r,k} & 0_{r_1 s\times pk(k-s)} \\ I_r \otimes (\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} \tag{79}$$

$$\times K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix} \tag{80}$$

$$= \begin{bmatrix} K_{r_1,s} \left(\Omega_{1\cdot}^{-1} \otimes \begin{bmatrix} I_{r_1} & 0_{r_1\times r_2} \end{bmatrix} \right) K_{r,k} & 0_{r_1 s\times pk(k-s)} \\ I_r \otimes (\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix}. \tag{81}$$

$$= \begin{bmatrix} \begin{bmatrix} I_{r_1} & 0_{r_1\times r_2} \end{bmatrix} \otimes \Omega_{1\cdot}^{-1} & 0_{r_1 s\times pk(k-s)} \\ I_r \otimes (\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)} \\ 0_{p(k-s)^2\times rk} & I_{p(k-s)} \otimes (\Omega^{-1})_{2\cdot} \end{bmatrix} K_{r+p(k-s),k} \operatorname{vec} \begin{bmatrix} \beta' \tilde{y}_{-1}' \tilde{\varepsilon} \\ \Delta Y' \tilde{\varepsilon} \end{bmatrix}. \tag{82}$$

Consequently, by Lemma 1

$$T^{-1/2} F_{0,\beta}(\beta)' \left(\Omega^{-1} \otimes I_{k+p(k-s)} \right) \operatorname{vec} \left(\tilde{Y}_{-1}' \tilde{\varepsilon} \right) \xrightarrow{d}$$

$$N_{r_1 s + r(k-s) + p(k-s)^2} \left(0, \begin{bmatrix} \Sigma_{\beta\beta,11} \otimes (\Omega^{-1})_{11} & \Sigma_{\beta\beta,1\cdot} \otimes (\Omega^{-1})_{12} & \Sigma_{\beta 0,1\cdot} \otimes (\Omega^{-1})_{12} \\ \Sigma_{\beta\beta,\cdot 1} \otimes (\Omega^{-1})_{21} & \Sigma_{\beta\beta} \otimes (\Omega^{-1})_{22} & \Sigma_{\beta 0} \otimes (\Omega^{-1})_{22} \\ \Sigma_{0\beta,\cdot 1} \otimes (\Omega^{-1})_{21} & \Sigma_{0\beta} \otimes (\Omega^{-1})_{22} & \Sigma_{00} \otimes (\Omega^{-1})_{22} \end{bmatrix} \right). \tag{83}$$

The asymptotic covariance matrix in (83) follows from $T^{-1/2}\operatorname{vec}\begin{bmatrix}\beta'\tilde{y}_{-1}'\tilde{\varepsilon}\\ \Delta Y'\tilde{\varepsilon}\end{bmatrix} \xrightarrow{d} \operatorname{vec}\begin{bmatrix}\zeta\\ \xi\end{bmatrix}$ in (82) by Lemma 1, where also the variance of $K_{r+p(k-s),k}\operatorname{vec}\begin{bmatrix}\zeta\\ \xi\end{bmatrix} = \operatorname{vec}\begin{bmatrix}\zeta' & \xi'\end{bmatrix}$ is $\Sigma\otimes\Omega$. The variance in the asymptotic distribution in (83) then comes from

$$\begin{bmatrix}\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix}\otimes\Omega_{1\cdot}^{-1} & 0_{r_1 s\times pk(k-s)}\\ I_r\otimes(\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)}\\ 0_{p(k-s)^2\times rk} & I_{p(k-s)}\otimes(\Omega^{-1})_{2\cdot}\end{bmatrix}\Sigma\otimes\Omega \tag{84}$$

$$\times\begin{bmatrix}\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix}\otimes\Omega_{1\cdot}^{-1} & 0_{r_1 s\times pk(k-s)}\\ I_r\otimes(\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)}\\ 0_{p(k-s)^2\times rk} & I_{p(k-s)}\otimes(\Omega^{-1})_{2\cdot}\end{bmatrix}' \tag{85}$$

$$=\begin{bmatrix}\Sigma_{\beta\beta,1\cdot}\otimes\begin{bmatrix}I_s & 0_{s\times k-s}\end{bmatrix} & \Sigma_{\beta 0,1\cdot}\otimes\begin{bmatrix}I_s & 0_{s\times k-s}\end{bmatrix}\\ \Sigma_{\beta\beta}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} & \Sigma_{\beta 0}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\\ \Sigma_{0\beta}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix} & \Sigma_{00}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\end{bmatrix} \tag{86}$$

$$\times\begin{bmatrix}\begin{bmatrix}I_{r_1} & 0_{r_1\times r_2}\end{bmatrix}\otimes\Omega_{1\cdot}^{-1} & 0_{r_1 s\times pk(k-s)}\\ I_r\otimes(\Omega^{-1})_{2\cdot} & 0_{r(k-s)\times pk(k-s)}\\ 0_{p(k-s)^2\times rk} & I_{p(k-s)}\otimes(\Omega^{-1})_{2\cdot}\end{bmatrix}' \tag{87}$$

$$=\begin{bmatrix}\Sigma_{\beta\beta,11}\otimes(\Omega^{-1})_{11} & \Sigma_{\beta\beta,1\cdot}\otimes(\Omega^{-1})_{12} & \Sigma_{\beta 0,1\cdot}\otimes(\Omega^{-1})_{12}\\ \Sigma_{\beta\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{\beta\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{\beta 0}\otimes(\Omega^{-1})_{22}\\ \Sigma_{0\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{0\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{00}\otimes(\Omega^{-1})_{22}\end{bmatrix}. \tag{88}$$

As the (scaled) estimator can be written as the inverse of (64) times (83), the asymptotic distribution of the estimator is

$$\sqrt{T}(\hat{\pi}_\alpha-\pi_\alpha)\xrightarrow{d} N_{r_1 s+r(k-s)+p(k-s)^2}(0,V) \tag{89}$$

where

$$V=\begin{bmatrix}\Sigma_{\beta\beta,11}\otimes(\Omega^{-1})_{11} & \Sigma_{\beta\beta,1\cdot}\otimes(\Omega^{-1})_{12} & \Sigma_{\beta 0,1\cdot}\otimes(\Omega^{-1})_{12}\\ \Sigma_{\beta\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{\beta\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{\beta 0}\otimes(\Omega^{-1})_{22}\\ \Sigma_{0\beta,\cdot 1}\otimes(\Omega^{-1})_{21} & \Sigma_{0\beta}\otimes(\Omega^{-1})_{22} & \Sigma_{00}\otimes(\Omega^{-1})_{22}\end{bmatrix}^{-1}. \tag{90}$$

C.2 Asymptotic Distribution of $\hat{\pi}_\beta$ Under Granger Non-Causality

Let

$$\beta=\left[\begin{array}{c|c}\beta_{11} & \beta_{12}\\ \hline \beta_{21}=0_{k-s\times r_1} & \beta_{22}\end{array}\right] \tag{91}$$

where β_{11} is $s\times r_1$, β_{21} is $k-s\times r_1$, β_{12} is $s\times r_2$ and β_{22} is $(k-s)\times r_2$. r_1 denotes cointegrating relations which enter the foreign variables. For convenience, let also $\beta_1=(\beta_{11},\beta_{12})$ and $\beta_2=(\beta_{21},\beta_{22})$.

The zero restrictions can be enforced by the following decomposition

$$\operatorname{vec}(\Pi)'=\begin{bmatrix}\operatorname{vec}(\alpha\beta')\\ \operatorname{vec}\begin{bmatrix}0\\ \Gamma_d\end{bmatrix}\end{bmatrix} \tag{92}$$

$$=\underbrace{\left[\begin{array}{c|c|c}\begin{bmatrix}I_s\\ 0_{k-s\times s}\end{bmatrix}\otimes\alpha & \begin{matrix}0_{ks\times r_2(k-s)}\\ I_{k-s}\otimes\alpha\begin{bmatrix}0_{r_1\times r_2}\\ I_{r_2}\end{bmatrix}\end{matrix} & 0_{r(k-s)\times p(k-s)^2}\\ \hline 0_{pk(k-s)\times rs} & 0_{pk(k-s)\times r_2(k-s)} & I_{p(k-s)}\otimes\begin{bmatrix}0_{s\times k-s}\\ I_{k-s}\end{bmatrix}\end{array}\right]}_{F_{2,\alpha}(\alpha)}\begin{bmatrix}\operatorname{vec}(\beta_1')\\ \operatorname{vec}(\beta_{22}')\\ \operatorname{vec}(\Gamma_d)\end{bmatrix}. \tag{93}$$

Let

$$F_{0,\alpha}(\alpha) = \left(I \otimes \tilde{Y}_{-1}'\tilde{Y}_{-1}\right) F_{1,\alpha}(\alpha) = \left(I \otimes \tilde{Y}_{-1}'\tilde{Y}_{-1}\right) K_{k,k+p(k-s)} F_{2,\alpha}(\alpha). \tag{94}$$

It is then possible to rewrite (19) as

$$\hat{\pi}_\beta = \pi_\beta + \left\{F_{1,\alpha}(\alpha)' \left[\Omega^{-1} \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right] F_{1,\alpha}(\alpha)\right\}^{-1} F_{1,\alpha}(\alpha)' \left(\Omega^{-1} \otimes I\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\tilde{\varepsilon}\right) \tag{95}$$

by using the identity $(I\otimes A)(B\otimes A^{-1})(I\otimes A) = (B\otimes A)$. Consider now $\Upsilon_T = \mathrm{diag}(T^{-1}I_{rs+r_2(k-s)}, T^{-1/2}I_{p(k-s)^2})$ and

$$\Upsilon_T F_{1,\alpha}(\alpha)' = \Upsilon_T F_{2,\alpha}(\alpha)' K_{k+p(k-s),k} = F_{2,\alpha}(\alpha)'(\ddot{\Upsilon}_T \otimes I_k) K_{k+p(k-s),k}. \tag{96}$$

Additionally, we can write

$$(\Omega^{-1} \otimes \tilde{Y}_{-1}'\tilde{Y}) = (I_k \otimes \tilde{Y}_{-1}')(\Omega^{-1} \otimes I_T)(I_k \otimes \tilde{Y}_{-1}). \tag{97}$$

By Magnus and Neudecker (1979, Theorem 3.1(viii)) we also have

$$K_{k+p(k-s),k}(I_k \otimes \tilde{Y}_{-1}') = (\tilde{Y}_{-1}' \otimes I_k) K_{T,k} \tag{98}$$

and hence it follows that

$$\begin{aligned}(\ddot{\Upsilon}_T \otimes I_k) K_{k+p(k-s),k}(I_k \otimes \tilde{Y}_{-1}') &= (\ddot{\Upsilon}_T \otimes I_k)(\tilde{Y}_{-1}' \otimes I_k) K_{T,k} \\ &= (\ddot{\Upsilon}_T \tilde{Y}_{-1}' \otimes I_k) K_{T,k}\end{aligned} \tag{99}$$

By symmetry, (96) and (99) imply that

$$\left\{\Upsilon_T F_{1,\alpha}(\alpha)' \left[\Omega^{-1} \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right] F_{1,\alpha}(\alpha) \Upsilon_T\right\}^{-1} \tag{100}$$

$$= \left\{F_{1,\alpha}(\alpha)' \left[\Omega^{-1} \otimes \left(\ddot{\Upsilon}_T \tilde{Y}_{-1}'\tilde{Y}_{-1}\ddot{\Upsilon}_T\right)\right] F_{1,\alpha}(\alpha)\right\}^{-1} \tag{101}$$

By Lemma 1 and the continuous mapping theorem, we obtain

$$\begin{aligned}&\left\{\Upsilon_T F_{1,\alpha}(\alpha)' \left[\Omega^{-1} \otimes \left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right] F_{1,\alpha}(\alpha) \Upsilon_T\right\}^{-1} \\ \xrightarrow{p} &\left\{F_{1,\alpha}(\alpha)' \left[\Omega^{-1} \otimes \begin{bmatrix} \int G_k(u)G_k(u)'du & 0_{k\times p(k-s)} \\ 0_{p(k-s)\times k} & \Sigma_{00}\end{bmatrix}\right] F_{1,\alpha}(\alpha)\right\}^{-1}\end{aligned} \tag{102}$$

Furthermore, we can write

$$\begin{aligned}&\Upsilon_T F_{1,\alpha}(\alpha)' \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\ = &F_{2,\alpha}(\alpha)'(\ddot{\Upsilon}_T \otimes I_k) K_{k+p(k-s),k} \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right),\end{aligned} \tag{103}$$

where also

$$\begin{aligned}&(\ddot{\Upsilon}_T \otimes I_k) K_{k+p(k-s),k} \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\ &= K_{k+p(k-s),k}(I_k \otimes \ddot{\Upsilon}_T) \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\ &= K_{k+p(k-s),k}(\Omega^{-1} \otimes \ddot{\Upsilon}_T) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\ &= K_{k+p(k-s),k} \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\ddot{\Upsilon}_T \tilde{Y}_{-1}'\varepsilon\right).\end{aligned} \tag{104}$$

Thus, we have by (103), (104) and the continuous mapping theorem

$$\begin{aligned}&\Upsilon_T F_{1,\alpha}(\alpha)' \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\ &\xrightarrow{d} F_{1,\alpha}(\alpha) \left(\Omega^{-1} \otimes I_{k+p(k-s)}\right) \mathrm{vec}\begin{bmatrix}\int G_k dW_k' \\ \xi\end{bmatrix}.\end{aligned} \tag{105}$$

Taking (102) and (105) together results in

$$\begin{aligned}
\Upsilon_T^{-1}\left(\hat{\pi}_\beta - \pi_\beta\right) &= \left\{\Upsilon_T F_{1,\alpha}(\alpha)'\left[\Omega^{-1}\otimes\left(\tilde{Y}_{-1}'\tilde{Y}_{-1}\right)\right]F_{1,\alpha}(\alpha)\Upsilon_T\right\}^{-1} \\
&\times \Upsilon_T F_{1,\alpha}(\alpha)'\left(\Omega^{-1}\otimes I_{k+p(k-s)}\right)\operatorname{vec}\left(\tilde{Y}_{-1}'\varepsilon\right) \\
&\xrightarrow{d} \left\{F_{1,\alpha}(\alpha)'\left[\Omega^{-1}\otimes\begin{bmatrix}\int G_k(u)G_k(u)'du & 0_{k\times p(k-s)} \\ 0_{p(k-s)\times k} & \Sigma\end{bmatrix}\right]F_{1,\alpha}(\alpha)\right\}^{-1} \\
&\times F_{1,\alpha}(\alpha)'\left(\Omega^{-1}\otimes I_{k+p(k-s)}\right)\operatorname{vec}\begin{bmatrix}\int G_k dW_k' \\ \xi\end{bmatrix}
\end{aligned}\tag{106}$$

Next,

$$\begin{aligned}
&F_{1,\alpha}(\alpha)'\left[\Omega^{-1}\otimes\begin{bmatrix}\int G_k(u)G_k(u)'du & 0_{k\times p(k-s)} \\ 0_{p(k-s)\times k} & \Sigma_{00}\end{bmatrix}\right]F_{1,\alpha}(\alpha) \\
&= F_{2,\alpha}(\alpha)'\left(\begin{bmatrix}\int G_k(u)G_k(u)'du & 0_{k\times p(k-s)} \\ 0_{p(k-s)\times k} & \Sigma_{00}\end{bmatrix}\otimes\Omega^{-1}\right)F_{2,\alpha}(\alpha) \\
&= F_{2,\alpha}(\alpha)'\begin{bmatrix}\left(\int G_k(u)G_k(u)'du\right)\otimes\Omega^{-1} & 0_{k^2\times pk(k-s)} \\ 0_{pk(k-s)\times k^2} & \Sigma_{00}\otimes\Omega^{-1}\end{bmatrix}F_{2,\alpha}(\alpha) \\
&= \\
&\begin{bmatrix}\int G_{k,1}(u)G_{k,1}(u)'du\otimes\alpha'\Omega^{-1}\alpha & \int G_{k,1}(u)G_{k,2}(u)'du\otimes\alpha'\Omega^{-1}\alpha_{\cdot 2} & 0_{rs\times pk(k-s)} \\ \int G_{k,2}(u)G_{k,1}(u)'du\otimes\alpha_{\cdot 2}'\Omega^{-1}\alpha & \int G_{k,2}(u)G_{k,2}(u)'du\otimes\alpha_{\cdot 2}'\Omega^{-1}\alpha_{\cdot 2} & 0_{rs\times pk(k-s)} \\ 0_{pk(k-s)\times rs} & 0_{pk(k-s)\times r(k-s)} & \Sigma_{00}\otimes(\Omega^{-1})_{22}\end{bmatrix}
\end{aligned}\tag{107}$$

Similarly,

$$F_{1,\alpha}(\alpha)'\left(\Omega^{-1}\otimes I_{k+p(k-s)}\right)\operatorname{vec}\begin{bmatrix}\int G_k dW_k' \\ \xi\end{bmatrix}\tag{108}$$

$$= F_{2,\alpha}(\alpha)'\operatorname{vec}\left[\Omega^{-1/2}\int dB_k G_k' \quad \Omega^{-1}\xi'\right]\tag{109}$$

$$= \begin{bmatrix}\begin{bmatrix}I_s & 0_{s\times k-s}\end{bmatrix}\otimes\alpha'\operatorname{vec}\left(\Omega^{-1/2}\int dB_k G_k'\right) \\ \begin{bmatrix}0_{r_2(k-s)\times ks} & I_{k-s}\otimes\alpha_{\cdot 2}'\end{bmatrix}\operatorname{vec}\left(\Omega^{-1/2}\int dB_k G_k'\right) \\ I_{p(k-s)}\otimes\begin{bmatrix}0_{k-s\times s} & I_{k-s}\end{bmatrix}\operatorname{vec}\left(\Omega^{-1}\xi'\right)\end{bmatrix}\tag{110}$$

$$= \begin{bmatrix}\operatorname{vec}\left(\alpha'\Omega^{-1/2}\int dB_k G_{k,1}'\right) \\ \operatorname{vec}\left(\alpha_{\cdot 2}'\Omega^{-1/2}\int dB_k G_{k,2}'\right) \\ \operatorname{vec}\left((\Omega^{-1})_{2\cdot}\xi'\right)\end{bmatrix}.\tag{111}$$

Thus,

$$\Upsilon_T^{-1}(\hat{\pi}_\beta - \pi_\beta)\xrightarrow{d}\tag{112}$$

$$\begin{bmatrix}\int G_{k,1}(u)G_{k,1}(u)'du\otimes\alpha'\Omega^{-1}\alpha & \int G_{k,1}(u)G_{k,2}(u)'du\otimes\alpha'\Omega^{-1}\alpha_{\cdot 2} & 0_{rs\times pk(k-s)} \\ \int G_{k,2}(u)G_{k,1}(u)'du\otimes\alpha_{\cdot 2}'\Omega^{-1}\alpha & \int G_{k,2}(u)G_{k,2}(u)'du\otimes\alpha_{\cdot 2}'\Omega^{-1}\alpha_{\cdot 2} & 0_{rs\times pk(k-s)} \\ 0_{pk(k-s)\times rs} & 0_{pk(k-s)\times r(k-s)} & \Sigma_{00}\otimes(\Omega^{-1})_{22}\end{bmatrix}^{-1}\tag{113}$$

$$\times\begin{bmatrix}\operatorname{vec}\left(\alpha'\Omega^{-1/2}\int dB_k G_{k,1}'\right) \\ \operatorname{vec}\left(\alpha_{\cdot 2}'\Omega^{-1/2}\int dB_k G_{k,2}'\right) \\ \operatorname{vec}\left((\Omega^{-1})_{2\cdot}\xi'\right)\end{bmatrix}\tag{114}$$